Gennadiy A. Kalyabin
Peoples Friendship University Russia, Moscow, Russia

Sharp Friedrichs and Poincaré inequalities for Sobolev spaces on finite interval

In [1] the constructive formulas were obtained for the sharp (i.e. the least possible) constants $A_{r,k}(x),\ r \in \mathbb{N},\ k \in \{0, 1, \ldots, r-1\}, x \in (-1, 1)$ in the Friedrichs-Kolmogorov type inequalities

$$|f^{(k)}(x)| \le A_{r,k}(x) \left(\int_{-1}^{1} |f^{(r)}(t)|^2 \, dt \right)^{1/2} ; \quad f \in C_0^\infty(-1, 1).$$

Theorem 1.

$$((r-k-1)!)^2 A_{r,k}^2(x) = \frac{(1-x^2)^{2r-2k-1}}{2^{2r-2k-1}(2r-2k-1)} - \sum_{n=r-k}^{r-1} (n+0.5) \left(\frac{D^{n-r+k}((x^2-1)^n)}{2^n n!} \right)^2 ; \;\; D := \frac{d}{dx}.$$

From here applying classical variational principle one easily comes to the following assertion:

Corollary. *For the first (i.e. the least) eigenvalue $\lambda_{1,r}$ of the boundary value problem:*

$$(-D^2)^r y = \lambda y, \; y^{(s)}(\pm 1) = 0, \; s \in \{0, 1, \ldots r-1\},$$

the asymptotic relationship $\lambda_{1,r} = \sqrt{2}\,(2r)!\,(1 + O(1/r)),\ r \to \infty$, holds.

Note that this formula with the remainder term $O(1/\sqrt{r})$ was firstly obtained in [2] by investigating special determinants behavior.
Quite new seems the problem of finding the sharp constants $B_m(x), m \in \mathbb{N}_0,\ x \in (-1, 1)$ in Poincaré type inequalities:

$$|f(x)| \le B_m(x) \Big(\int_{-1}^{1} |f'(t)|^2 dt \Big)^{1/2} ; \; f \in W_2^1, \; \int_{-1}^{1} f(t) t^j \, dt = 0, \; j \in \{0, \ldots, m\}.$$

Partially the answer is given by

Theorem 2. $B_m^2(1) = B_m^2(-1) = 2B_m^2(0) = \frac{2}{(m+1)(m+3)}$.

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References

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[2] *Böttcher A., Widom H.* From Töplitz eigenvalues through Green's kernels to higher-order Wirtinger-Sobolev inequalities. *The extended field of operator theory*, 73–87, *Oper. Theory Adv. Appl.*, *Birkhäuser*, Basel, 2007.