Algorithm of the decision of equation Fredholm with a kernel and free member from a classes of Korobov

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12.10.2010

Let number $s\,(s = 1, 2, \ldots)$ is given. Let functions $K(x, y)$ and $f(x)$ are defined and continuous on $[0, 1]^{2s}$ and $[0, 1]^{s}$, respectively.

The equation

$$g(x) - \lambda \int_{[0,1]^s} K(x, y)\, g(y)\, dy = f(x) \qquad (1)$$

N.M.Korobov (see [3,p. 203-213], and also [4, p. 186-195]) the method *"data of the integrated equation to system of the algebraic equations by application quadrature formulas with number-theoretic grids"*, receives the approached decision of the equation (1) under conditions of an accessory of kernel $K(x,y)$ and of free member $f(x)$ to classes E_{2s}^{r} and E_{s}^{r}, respectively.

Definition

Class Korobov $E_t^r \quad (t = 1, 2, ...; r > 1)$ consists of all 1-periodic on each of t variable functions $f\left(x_1, \ldots, x_t\right) \in L\left(0, 1\right)^t$, trigonometrical coefficients Fourier which satisfy to a condition

$$\left|\widehat{f}\left(m\right)\right| \leq \left(\bar{m}_1 \ldots \bar{m}_t\right)^{-r},$$

where $\bar{m}_j = max\left\{1, \left|m_j\right|\right\}$

However, the received N.M.Korobov result has character of the theorem of existence: *"if* $a_1, \ldots, a_s$ *- optimal coefficients..."*

The given report is devoted the equation (1) in the conditions of resulted above works N.M.Korobov added with full algorithm of construction of the decision.

Let's result necessary for the further definition and a designation. Through $D(\lambda)$ designate *Fredholm determinant* of kernel $K(x, y)$:

$$D(\lambda) = 1 + \sum_{v=1}^{\infty} \frac{(-1)^v}{v!} \lambda^v \int\limits_{[0,1]^{sv}} K\begin{pmatrix} P_1, & ..., & P_v \\ P_1, & ..., & P_v \end{pmatrix} dP_1 ... dP_v$$

where

$$K\begin{pmatrix} P_1, & ..., & P_v \\ Q_1, & ..., & Q_v \end{pmatrix} = \begin{vmatrix} K(P_1, Q_1) & \cdots & K(P_1, Q_v) \\ \cdots & \cdots & \cdots \\ K(P_v, Q_1) & \cdots & K(P_v, Q_v) \end{vmatrix}$$

Some data from the algebraic theory of numbers (see, for example, [5]) also be required to us.

Let $l \geq 3$- prime number, $\theta = cos\frac{2\pi}{l} + isin\frac{2\pi}{l} = e^{i2\pi/l}$ Denote by $Q(\theta)$ the subfield of the complex number field that consists of all numbers α of the form

$$\alpha = c_1 + c_2\theta + ... + c_{l-1}\theta^{l-2},$$

where $c_1, c_2, ..., c_{l-1}$- are arbitrary rational numbers. The multiplication of two numbers from $Q(\theta)$ is performed in the usual manner by using the commutative, associative, and distributive laws and the equalities

$$\theta^{k+l} = \theta^k, \theta^{l-1} = -1 - \theta - ... - \theta^{l-2}.$$

If we set $s = l - 1$, then the numbers $\omega_1 = 1, \omega_2 = \theta, ..., \omega_s = \theta^{s-1}$ form fundamental basis of $Q(\theta)$ the set A_s of all integer algebraic numbers of $Q(\theta)$ is a ring and any number $m \in A_s$ can be uniquely represented as $m = m_1\omega_1 + m_2\omega_2 + ... + m_s\omega_s$ where $m_1, m_2, ..., m_s$ are rational integers.

We identify Z^s with A_s (and, hence, their subsets) $m = (m_1, m_2, ..., m_s) \in Z^s \leftrightarrow m = m_1\omega_1 + m_2\omega_2 + ... + m_s\omega_s \in A_s$.

A nonempty subset $\Re$ of A_s is called as an ideal in A_s if, given a and b that are in $\Re$, $ma + nb$ is also in $\Re$ for arbitrary m and n from A_s. Obvious, A_s is itself an ideal. The ideal A_s is called a unit ideal and is denoted by (1). If $m \in A_s$, it is easy to see that $ma : a \in A_s$ an ideal. It is denoted by (m) and is called the principal ideal generated by m. Let $\Re_s$ be an ideal and $\gamma_1, \gamma_2, ..., \gamma_s$ be a basis of $\Re$ such that

$$\gamma_k = \sum_{j=1}^{s} c_{k,j}\omega_j \ (k = 1, 2, ..., s).$$

$N\Re = \left| det\left(c_{k_j}\right) \right|$ is integer rational and is called the norm of $\Re$. For the principal ideal $\Re = (m)$ we have

$$N(m) = \prod_{k=1}^{s} \left(m_1 + m_2\theta^k + ... + m_s\theta^{(s-1)k}\right).$$

[5, p. 102, theorem 76]. For a bounded set $E \subset Z^s$ define

$$K(E) = \prod_{m \in E^*} N(m).$$

Theorem. Let are given prime number $l \geq 3$, $l = s + 1$ and $r > 2$.
Let $K(x_1, ..., x_s, y_1, ..., y_s) \in E^r_{2s}$,
f(x$_1, ..., x_s) \in E^r_s$.
and let λ such, that $D(\lambda) \neq 0$. Let it is given $T > 0$. Let's execute following actions:
1^o. We will put
$E = \Gamma_R \equiv \{m = (m_1, m_2, ..., m_s) \in Z^s : \bar{m}_1 ... \bar{m}_s \leq R\}$ where R such, that $c(s) R ln^s R = T$;
2^o. According to algorithm

(i) Is $K(E) = \prod\limits_{m \in E} N(m)$;

(ii) All prime numbers from interval $(1, 18 s ln K(E))$ are a method Sieve of Eratosthenes;

(iii) Direct check of everyone prime p ,
$p \equiv 1 (mod l)$ $p \in (1, 18 s K(E))$is such which does not divide $K(E)$;

(iv) Find an integer a such, that $a^{\frac{p-1}{l}} \neq 1 (mod p)$ For $<< T ln ln T$ elementary arithmetic operations are prime number p.

3^o. Grid $\eta_k(a) = \left(\frac{k}{p}, \left\{\frac{k}{p}a^{\frac{p-1}{l}}\right\}, ..., \left\{\frac{k}{p}a^{\frac{p-1}{l}(s-1)}\right\}\right) \quad (k = 1, ..., p)$
leaves;
4^o. Equation $\tilde{g}\left(\eta_j(a)\right) =$
$\frac{\lambda}{p}\sum\limits_{k=1}^{p} K(\eta_j(a), \eta_k(a))\tilde{g}\left(\eta_k(a)\right) + f(\eta_j(a)) \quad (j = 1, 2, ..., p)$
solves;
5^o. There is approached decision $g(x)$ equation (1):
$g(x) = \frac{\lambda}{p}\sum\limits_{k=1}^{p} K(x, \eta_k(a))\tilde{g}\left(\eta_k(a)\right) + f(x) + O(T^{-r}(lnT)^{3rs})$.

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