On error bounds for L_∞-approximation of smooth functions

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SAFS'10 - International Workshop on Smoothness, Approximation and Function Spaces, Oppurg
October 14th, 2010

Some motivation

In the 2nd talk today we saw that for the approximation of d-variate, smooth functions in the norm of L_∞ we need an exponential amount of information $n(\varepsilon, d)$ to obtain a worst case error $\varepsilon = e(n, d) < 1$.

Some motivation

In the 2nd talk today we saw that for the approximation of d-variate, smooth functions in the norm of L_∞ we need an exponential amount of information $n(\varepsilon, d)$ to obtain a worst case error $\varepsilon = e(n, d) < 1$.
In detail, we have

$$e(n, d) = 1 \qquad \forall n < 2^{\lfloor d/2 \rfloor} \quad \forall d \in \mathbb{N},$$

or

$$n(\varepsilon, d) \geq 2^{\lfloor d/2 \rfloor} \qquad \forall \varepsilon \in (0, 1) \quad \forall d \in \mathbb{N}.$$

So the problem suffers from the so-called *curse of dimensionality* and is *intractable*. See Novak & Woźniakowski (2009).

Therefore, we introduce **weights** in order to shrink the function space and break this exponential dependence on the dimension d. In the case of Hilbert spaces this idea goes back to Sloan and Woźniakowski (1998). Additionally, for Banach spaces discussed here we need essentially new proof techniques.

We will present a lower bound result which relates the worst case error to the used weights and show its application on important examples.

For simplicity we restrict ourself to the easiest case in this talk.

Overview

1. Introduction
   - The weighted approximation problem
   - An error criterion
   - Notions of tractability
2. The lower bound result
3. Applications / examples
   - Unweighted case
   - Finite-order weights
   - Product weights
4. Final remarks

The weighted approximation problem

Let $d \in \mathbb{N}$ (dimension), as well as $\Omega = [0,1]^d$ and

$$F_d^\gamma(\Omega) := \{ f \colon \Omega \to \mathbb{R} \mid f \in C^{(1,\ldots,1)}(\Omega), \, \|f\|_\gamma < \infty \},$$
$$\|f\|_\gamma := \max_{\alpha \in \{0,1\}^d} \frac{1}{\gamma_\alpha} \|D^\alpha f\|_\infty$$

with a fixed weight $\gamma = (\gamma_\alpha)_{\alpha \in \{0,1\}^d}$ where $\gamma_\alpha \geq 0$ and $\gamma_0 = 1$.

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For every $d \in \mathbb{N}$ approximate

$$\mathrm{Id}_d \colon F_d^\gamma(\Omega) \to L_\infty(\Omega), \quad \mathrm{Id}_d(f) := f$$

by operators $S_{n,d} = \phi \circ N$ using $n \in \mathbb{N}_0$ pieces of information from $f \in F_d^\gamma$,

$$N \colon F_d^\gamma \to \mathbb{R}^n, \qquad \phi \colon \mathbb{R}^n \to L_\infty.$$

Note that

- F_d^γ is an infinite dimensional Banach space, mainly characterized by the weights γ,
- $\|f\|_\gamma \leq 1 \quad \Longleftrightarrow \quad \|D^\alpha f\|_\infty \leq \gamma_\alpha$ for all $\alpha \in \{0,1\}^d$ ("$\frac{0}{0} := 0$"),

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- $N = N_{n,d}$ collects the *information*; should be a continuous mapping (e.g. linear functionals / functions values)
- $\phi = \phi_{n,d}$ creates the approximation; can be chosen arbitrarily

$\Rightarrow$ different classes of weights & different types of algorithms possible

'Good' approximation?!

(Absolute) *worst case error* for algorithm $S_{n,d}$:

$$e^{\mathrm{wor}}(S_{n,d}) := \sup_{\substack{f \in F_d^\gamma \\ \|f\|_\gamma \leq 1}} \left\| \mathrm{Id}_d(f) - S_{n,d}(f) \,|\, L_\infty(\Omega) \right\| .$$

n-th minimal error in dimension d:

$$e(n,d) := \inf_{S_{n,d} \in \Lambda_{n,d}} e^{\mathrm{wor}}(S_{n,d}).$$

Information complexity:

$$n(\varepsilon, d) := \inf\{ n \in \mathbb{N}_0 \,|\, e(n,d) \leq \varepsilon \}, \quad \varepsilon > 0.$$

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Note that $S_{0,d} := 0 \in L_\infty$. Therefore, for every $n \in \mathbb{N}_0$ and $d \in \mathbb{N}$ we have the trivial upper bound

$$e(n,d) \leq e^{\mathrm{wor}}(S_{0,d}) = \left\| \mathrm{Id}_d \colon F_d^\gamma \to L_\infty \right\| = \sup_{\|f\|_\gamma \leq 1} \|f\|_\infty = \gamma_0 = 1.$$

Classes of tractability

The problem is called

- *strongly polynomially tractable* (SPT), iff

$$\exists C, p > 0 \text{ s.t. } \forall d \in \mathbb{N}\ \forall \varepsilon \in (0,1) : \quad n(\varepsilon, d) \leq C \cdot \varepsilon^{-p}$$

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Obviously,

$$\text{SPT} \Longrightarrow \text{PT} \Longrightarrow \text{WT} \Longrightarrow \text{no COD}.$$

Theorem

Assume $d \in \mathbb{N}$, $\omega \in \mathbb{N}_0$, $\lambda > 0$ and let γ be a weight with

$$\gamma_\alpha \geq \lambda^{|\alpha|} \quad \textit{if} \quad \alpha \in \{0,1\}^d \quad \textit{and} \quad |\alpha| \leq \omega.$$

Then for the n-th minimal error of L_∞-approximation on $F_d^\gamma([0,1]^d)$ we have

$$e(n,d) \geq 1 \quad \textit{for all} \quad n < \sum_{m=0}^{\min\{\omega,d\}} \binom{\lfloor d/l \rfloor}{m},$$

where $l = \lceil 2/\lambda \rceil$.

1) Unweighted case

For $\gamma_\alpha \equiv 1$ the problem is unweighted. We set $\lambda := 1$ and $\omega := d$.

Then we have the **curse of dimensionality**, because

$$e(n,d) = 1 \quad \text{for all} \quad n < \sum_{m=0}^{d} \binom{\lfloor d/2 \rfloor}{m} = 2^{\lfloor d/2 \rfloor}.$$

Therefore, the theorem generalizes the results known before.
(In fact Novak & Woźniakowski considered an even smaller space of C^∞-functions but this doesn't matter)

2) Finite-order weights

Suppose the weights γ fulfill a *finite-order property*, i.e.

$$|\alpha| > \omega \quad \Longrightarrow \quad \gamma_\alpha = 0,$$

for a fixed $\omega < d$.
In this case we have for $f \in F_d^\gamma(\Omega)$ the representation

$$f = \sum_{\substack{\mathfrak{u} \subset \{1,\ldots,d\}, \\ \#\mathfrak{u} \leq \omega}} f_\mathfrak{u},$$

where $f_\mathfrak{u}$ only depends on at most $\#\mathfrak{u} \leq \omega$ variables. Applications can be found in physics, e.g. Coulomb potential.

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where $f_{\mathfrak{u}}$ only depends on at most $\#\mathfrak{u} \leq \omega$ variables. Applications can be found in physics, e.g. Coulomb potential.
The theorem yields

$$e(n,d) = 1 \quad \text{for all} \quad n < \sum_{m=0}^{\omega} \binom{\lfloor d/l \rfloor}{m} \sim d^\omega,$$

where the constants depend on γ but not on d. Hence, we have **no strong polynomial tractability**.

3) Product weights

Assume the weights γ have a product structure, i.e.

$$\gamma_\alpha = \prod_{k=1}^{d} (\gamma_{d,k})^{\alpha_k}, \quad \alpha \in \{0,1\}^d$$

with generators

$$1 \geq \gamma_{d,1} \geq \gamma_{d,2} \geq \ldots \geq \gamma_{d,d} > 0.$$

Here $\gamma_{d,k}$ moderates the influence of x_k.

A result similar to the mentioned theorem, some additional calculations and upper error bound results (due to Kuo, Wasilkowski and Woźniakowski) lead to:

Corollary

For the information complexity of L_∞-approximation on $F_d^\gamma(\Omega)$ in the case of product weights we have

$$n(\varepsilon, d) \geq 2^{\left\lfloor \frac{1}{3} \sum_{k=1}^{d} \gamma_{d,k} \right\rfloor},$$

for all $d \in \mathbb{N}$ and $\varepsilon \in (0,1)$.
Moreover, the following statements are equivalent:

- *The problem is weakly tractable.*
- *The problem does not suffer from the curse of dimensionality.*
- *There exists $t \in (0,1)$ such that*

$$\lim_{d\to\infty} \frac{1}{d} \sum_{k=1}^{d} \gamma_{d,k}^{t} = 0.$$

Final remarks

In general the absence of the curse of dimensionality does NOT imply weak tractability!

The last condition is a typical characterization of weak tractability for problems on Hilbert spaces.

Note that everything also works in a more general setting.

SUMMARY:

- **unweighted** case $\Longrightarrow$ **curse of dimensionality** (even for C^∞-functions)
- **weighted** case $\Longrightarrow$ different types of **tractability** (depending on the weights)

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Thank you for your attention!