Average best m-term approximation

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Outline

- Introduction, motivation and notation
  - Best m-term approximation
  - Models of noise and signals
  - $p = 2$
  - Definition of average best m-term appr.

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- Cone measure
- Surface measure
- Tensor product measures
- Numerics, graphs, literature

Best m-term approximation

$m \in \mathbb{N}_0$:

$$\Sigma_m := \Big\{ x = \{x_j\}_{j=1}^{\infty} : \#\{j \in \mathbb{N} : x_j \neq 0\} \leq m \Big\}$$

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$0 < p \leq q \leq \infty$: *best m-term approximation widths*

$$\sigma_m^{p,q} := \sup_{x : \|x\|_p \leq 1} \sigma_m(x)_q$$

$$2^{-1/p}(m+1)^{1/q-1/p} \leq \sigma_m^{p,q} \leq (m+1)^{1/q-1/p}, \quad m = 0, 1, 2, \dots$$

Applications of best m-term appr.

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Signal processing: lossy approximation techniques, especially MP3, JPEG, JPEG-2000 and some video formats

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Benchmark for algorithms and a theoretical tool in approximation theory:
Temlyakov, Kashin, Oswald and many others

Use of random structures in discrete mathematics:
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Given (fixed) input is processed with randomized algorithm and gives (with *high probability*) the right output

Compressed Sensing, Monte Carlo,
Approximate Nearest Neighbor

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Random signals could be used to test the speed of algorithms

Models of noise

White noise in signal processing (sound, music, video, ...): $\varepsilon\omega$

$\omega = (\omega_1, \dots, \omega_n) \in \mathbb{R}^n$ - independent (normal) Gaussian variables
$\varepsilon \cdot \|\omega\|_2$ - energy level of the noise

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spectral density is uniformly spread over the visible spectrum

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To test an algorithm: generate white noise many times...

Infinite-dimensional models of white noise?

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But it has almost surely $\|\omega\|_2 = \infty$!

Therefore, one usually chooses n large, but finite

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All signals have something in common - they are usually **sparse** (or nearly sparse) in a basis adapted to the specific area: Fourier series, Gabor frames, wavelets, discrete cosine transform, curvelets and other —lets, ('95 - Donoho, Candés, Johnstone, . . .)

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We are looking for

- a random model in $\mathbb{R}^n$
- the *typical vectors* are nearly sparse
- should depend on several parameters

Bernoulli-Gaussian model for sparse vectors

Let $x \in \mathbb{R}^n$ with

$$x_i = \varepsilon \varrho_i \omega_i, \quad i = 1, \dots, n,$$

where $\varepsilon > 0$,

$$\varrho_i = \begin{cases} 1 & \text{with prob. } p, \\ 0 & \text{with prob. } 1-p, \end{cases}$$

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The number of non-zero components is concentrated around $k := pn \ll n$
Badly concentrated for small k's, two "noisy" for large k's

Although x is sparse, it is rather a *randomly filtered noise* then a signal

The linear algebraic equation

$$Ax = y, \quad A \in \mathbb{R}^{n \times n}, \quad y \in \mathbb{R}^n$$

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The methods of compressed sensing extend naturally to $\ell_p^n, p < 1$!

Are vectors from the unit ball of ℓ_p^n really *nearly sparse*?

How does a typical vector of the ℓ_p^n unit ball look like?

or:

Let μ be a probability measure on the unit ball of ℓ_p^n. What is the mean value of $\sigma_m(x)_q$ with respect to this measure?

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Choice of μ?

(i) the normalized Lebesgue measure,

(ii) the $n-1$ dimensional Hausdorff measure restricted to the surface of the unit ball of ℓ_p^n and correspondingly normalized,

(iii) the so-called normalized cone measure

Well known case: $p = 2$, $q = \infty$, $m = 0$
μ . . . normalized surface measure on $\mathbb{S}^{n-1}$

....i.e. we want to calculate

$$\int_{\mathbb{S}^{n-1}} \max_{j=1,\dots,n} |x_j| d\mu(x) = \int_{\mathbb{S}^{n-1}} \|x\|_\infty d\mu(x) = \int_{\mathbb{S}^{n-1}} \sigma_0(x)_\infty d\mu(x)$$

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γ_n ... the standard Gaussian measure on $\mathbb{R}^n$ with the density

$$\frac{1}{(2\pi)^{n/2}} e^{-\|x\|_2^2/2}, \quad x \in \mathbb{R}^n$$

Polar coordinates: ($\Omega_n =$ the area of $\mathbb{S}^{n-1}$)

$$\int_{\mathbb{R}^n} \max_{j=1,\dots,n} |x_j| \, d\gamma_n(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \max_{j=1,\dots,n} |x_j| \cdot e^{-\|x\|_2^2/2} dx$$

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$$\int_{\mathbb{S}^{n-1}} \sigma_0(x)_\infty d\mu(x) \approx \sqrt{\frac{\log(n+1)}{n}}, \quad n \in \mathbb{N}$$

Interpretation:

The average coordinate of $x \in \mathbb{S}^{n-1}$: $= n^{-1/2}$
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Surprisingly, the same holds true for all $0 < p$!

Definitions: $0 < p \leq q \leq \infty$, $m \leq n$:

$$\Delta_p^n = \{(t_1, \ldots, t_n) \in \mathbb{R}_+^n : \sum_{j=1}^{n} t_j^p = 1\}$$

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$\mu \ldots$ a Borel probability measure on Δ_p^n

$$\sigma_m^{p,q}(\mu) = \int_{\Delta_p^n} \sigma_m(x)_q \, d\mu(x)$$

average surface best m-term width with respect to μ

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$\nu \ldots$ a Borel probability measure on $[0,1] \cdot \Delta_p^n$

$$\sigma_m^{p,q}(\nu) = \int_{\Delta_p^n} \sigma_m(x)_q \, d\nu(x)$$

average volume best m-term width with respect to ν

Cone measure

$n \geq 2$:

$$\mu_p(\mathcal{A}) = \frac{\lambda([0,1] \cdot \mathcal{A})}{\lambda([0,1] \cdot \Delta_p^n)}, \quad \mathcal{A} \subset \Delta_p^n$$

is the normalized *cone measure* on Δ_p^n

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polar decomposition identity:

$$\frac{\int_{\mathbb{R}_+^n} f(x)\,d\lambda(x)}{\lambda([0,1]\cdot\Delta_p^n)} = n\int_0^\infty r^{n-1}\int_{\Delta_p^n} f(rx)\,d\mu_p(x)\,dr, \quad f \in L_1(\mathbb{R}_+^n)$$

. . . substitute for *polar coordinates*

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...substitute for *polar coordinates*

$p = 1, 2, \infty$ ...coincides with the surface measure

Let $p = 2$ and let $\omega = (\omega_1, \ldots, \omega_n)$ be indep. Gaussian variables. Then

$$\mathbb{P}\left(\frac{(\omega_1, \ldots, \omega_n)}{\|\omega\|_2} \in \mathcal{A}\right) = \mu(A)$$

Proof: both the measures are rotational invariant on $\mathbb{S}^{n-1}$

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Variables with density $c_p e^{-t^p}$, $t > 0$, may be used if $p \neq 2$ (Schechtman and Zinn, 1990)

$$\mathbb{P}\left(\frac{(\omega_1, \dots, \omega_n)}{\|\omega\|_p} \in \mathcal{A}\right) = \mu_p(A)$$

Results (V., 2010):

(i) $n \geq 2$, $0 \leq m \leq n-1$, $0 < p < \infty$:

$$\sigma_m^{p,\infty}(\mu_p) \lesssim \left[\frac{\log\left(\frac{en}{m+1}\right)}{n}\right]^{1/p};$$

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Lebesgue measure

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The identity

$$\sigma_m^{p,q}(\nu_p) = \sigma_m^{p,q}(\mu_p) \cdot \frac{n}{n+1}$$

holds for all $0 < p \leq q \leq \infty$, all $n \geq 2$ and all $0 \leq m \leq n-1$

Surface measure

$n \geq 2$:

$$\varrho_p(\mathcal{A}) = \frac{\mathcal{H}(\mathcal{A})}{\mathcal{H}(\Delta_p^n)}, \qquad \mathcal{A} \subset \Delta_p^n$$

the normalized $n-1$ dimensional Hausdorff measure on Δ_p^n

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(ii) Density of ϱ_p w.r.t. μ_p
($p \geq 1$: Naor and Romik, 2003; $p < 1$, V., 2010)

$$\frac{d\varrho_p}{d\mu_p}(x) = \frac{n\lambda([0,1]\cdot\Delta_p^n)}{\mathcal{H}(\Delta_p^n)}\Big\|\nabla(\|\cdot\|_p)(x)\Big\|_2 = C_{p,n}^{-1}\left(\sum_{i=1}^n x_i^{2p-2}\right)^{1/2},$$

where $C_{p,n}$ is the normalizing constant

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Result (V. 2010): $n \geq 2$, $0 < p \leq \infty$:

$$\sigma_0^{p,\infty}(\varrho_p) \lesssim \left[\frac{\log(n+1)}{n} \right]^{1/p}$$

Tensor products

$0 < p < \infty$, $\beta > -1$: We define $\theta_{p,\beta}$ by

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$$C_{p,\beta} = \int_{\Delta_p^n} \prod_{i=1}^{n} x_i^{\beta} d\mu_p(x) \ldots \text{normalizing constant}$$

$\beta > -1$ implies $C_{p,\beta} < \infty$

Result: For every fixed $m \in \mathbb{N}$,

$$\frac{1}{\left(\frac{1}{p}+1\right)^m} \lesssim \liminf_{n\to\infty} \sigma^{p,\infty}_{m-1}(\theta_{p,p/n-1})$$

$$\leq \limsup_{n\to\infty} \sigma^{p,\infty}_{m-1}(\theta_{p,p/n-1}) \lesssim \frac{1}{\left(\frac{1}{p}+1\right)^m} + \frac{e^{-m}}{m!},$$

Interpretation:

The measure $\theta_{p,p/n-1}$ is strongly concentrated on nearly-sparse vectors. A typical vector with respect to $\theta_{p,p/n-1}$ *corresponds* to a structured signal

Can we take the limit $n \to \infty$: extension to function spaces?

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Potentially, one could prove theorems like: *"An algorithm is good for* `almost all` *pictures chosen w.r.t.* $\theta_{p,\beta}$ *..."*

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$$\mathbb{P}\left(\frac{(\omega_1',\dots,\omega_n')}{\|\omega'\|_p}\in\mathcal{A}\right)=\theta_{p,\beta}(\mathcal{A})$$

We take

$$\frac{(\omega_1',\dots,\omega_n')}{\left(\sum_{j=1}^n(\omega_j')^p\right)^{1/p}}\in\Delta_p^n.$$

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Box-Muller method:

$$X = \sqrt{-2 \ln U} \cos(2\pi V), \qquad Y = \sqrt{-2 \ln U} \sin(2\pi V)$$

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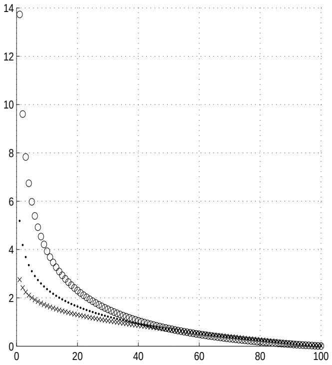
$$X=\sqrt{-2\ln U}\cos(2\pi V), \qquad Y=\sqrt{-2\ln U}\sin(2\pi V)$$

Marsaglia polar method

$$S := U^2+V^2<1, \quad X=U\sqrt{\frac{-2\ln S}{S}}, \quad Y=V\sqrt{\frac{-2\ln S}{S}}$$

...described in detail in Knuth and implemented in *GNU Scientific Library* for general p and β



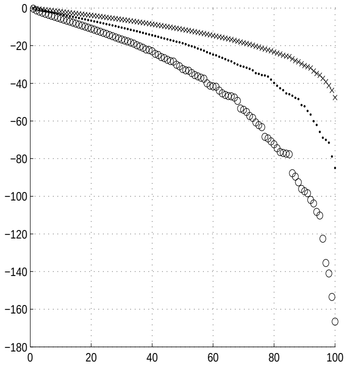


Figure: Approximations of $n^{1/p} \cdot \int_{\Delta_p^n} x_m^* d\mu_p(x)$ (left) and $\log_{10}(\int_{\Delta_p^n} x_m^* d\theta_{p,p/n-1})$ (right) for $n = 100$, $p = 1/2(\circ)$, $p = 1(\bullet)$ and $p = 2(\times)$ based on sampling of 10^8 random points.

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