Hardy inequalities, entropy numbers, spectral theory

Hans Triebel

University of Jena

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$B = \{x \in \mathbb{R}^n : \ |x| < 1\}$ unit ball in $\mathbb{R}^n$, $n \geq 2$.

$$E_b(f,g) = \int_B b(x) \sum_{j=1}^n \frac{\partial f(x)}{\partial x_j} \frac{\partial \bar{g}(x)}{\partial x_j} \, \mathrm{d}x, \qquad f,g \in D(B),$$

$$\|f \,|E_b(B)\| = E_b(f,f)^{1/2},$$

$0 < b(x) \in L_1^{\mathrm{loc}}(B)$, $E_b(B)$ completion, closable.

$$\mathrm{id}: \ E_b(B) \hookrightarrow L_2(B) \quad \text{if} \quad b^{-1} \in L_{n/2}(B),$$

positive definite quadratic form. Compact embedding? Zygmund spaces $L_p(\log L)_a(B)$, $0 < p < \infty$, $a \in \mathbb{R}$,

$$\|f \,|L_p(\log L)_a(B)\| = \Big(\int_0^{|B|} f^*(t)^p \big(1 + |\log t|\big)^{ap} \, \mathrm{d}t \Big)^{1/p}.$$

Here $f^*(t)$ (Lebesgue) measure-preserving decreasing rearrangement of f. If $a = 0$ then

$$L_p(\log L)_0(B) = L_p(B), \quad \|f \,|L_p(B)\| = \Big(\int_B |f(x)|^p \mathrm{d}x \Big)^{1/p}.$$

$e_k(\mathrm{id})$ entropy numbers defined later on.

Theorem 1. Let $n \geq 3$ and $b^{-1} \in L_{n/2}(\log L)_d(B)$ with $d > 0$. Then id is compact.

(i) If $d > 4/n$ then

$$e_k(\mathrm{id}) \leq c \,\| b^{-1} \,| L_{n/2}(\log L)_d(B)\|^{1/2} k^{-1/n}, \qquad k \in \mathbb{N}.$$

(ii) If $0 < d \leq 4/n$, then for any $\varepsilon > 0$ there is an $c_\varepsilon > 0$ such that

$$e_k(\mathrm{id}) \leq c_\varepsilon \,\| b^{-1} \,| L_{n/2}(\log L)_d(B)\|^{1/2} \, k^{-\frac{d}{4}+\varepsilon}, \qquad k \in \mathbb{N}.$$

Remark 2. Modification if $n = 2$.

If $\mathrm{id}: \; E_b(B) \hookrightarrow L_2(B)$ is compact then the quadratic form $E_b(\cdot,\cdot)$ generates a positive definite self-adjoint operator A_b in $L_2(B)$ with pure point spectrum $\{\lambda_k(A_b)\}_{k=1}^{\infty}$,

$$A_b f = - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \Big(b(\cdot) \, \frac{\partial f}{\partial x_j} \Big), \qquad f \in \mathrm{dom}\; A_b,$$

$$\mathrm{dom}\; \sqrt{A_b} = E_b(B) \qquad \text{energy space},$$

$$0 < \lambda_1(A_b) \le \lambda_2(A_b) \le \cdots \le \lambda_k(A_b) \le \cdots \to \infty \quad \text{if } k \to \infty,$$

$$a_k(\mathrm{id}) \sim \lambda_k^{-1/2}(A_b) \le c\, e_k(\mathrm{id}), \qquad k \in \mathbb{N}.$$

$a_k(\mathrm{id})$ approximation numbers of id, $e_k(\mathrm{id})$ entropy numbers.

Corollary 3. $n \ge 3$. $0 < b \in L_1^{\mathrm{loc}}(B)$, $b^{-1} \in L_{n/2}(\log L)_d(B)$. Then there are constants $c_0 > 0$, $c > 0$, $c_\varepsilon > 0$ for any $\varepsilon > 0$,

$$c_0 k^{2/n} \ge \lambda_k(A_b) \ge \begin{cases} c k^{2/n} & \text{if } d > 4/n, \\ c_\varepsilon k^{\frac{d}{2} - \varepsilon} & \text{if } 0 < d \le 4/n. \end{cases}$$

Remark 4. Modification if $n = 2$.

Breaking point for d in Theorem 1, Corollary 3? Candidate $d = 4/n$ (if $n \geq 3$). May depend on additional properties of b. Distinguished examples:

$$b(x) = |x|^2 \big(1 - \log |x|\big)^{2\sigma}, \qquad |x| < 1.$$

Proposition 5. $1 \leq p < \infty$, $\sigma \in \mathbb{R}$, $m \in \mathbb{N}$,

$$b(x) = |x|^{pm} \big(1 - \log |x|\big)^{\sigma p}, \qquad |x| < 1.$$

Then

$$b^{-1} \in L_{\frac{n}{pm}}(\log L)_d(B) \quad \text{if, and only if,} \quad \sigma > \frac{d}{p} + \frac{m}{n}.$$

Remark 6. $p = 2$, $m = 1$. Then

$$b^{-1} \in L_{n/2}(\log L)_d(B) \quad \text{if, and only if,} \quad \sigma > \frac{d}{2} + \frac{1}{n}.$$

Problem 7. One has by direct arguments

$$\mathrm{id}: \; E_b(B) \hookrightarrow L_2(B), \quad \text{with} \quad b(x) = |x|^2 \big(1 - \log |x|\big)^{2\sigma}, \; |x| < 1,$$

if, and only if, $\sigma \geq 0$. Furthermore, id is compact if, and only if, $\sigma > 0$. In the general approach (Theorem 1, Corollary 3), one needs $d > 0$, hence $\sigma > 1/n$.
Motivation: What happens in this special case with specific arguments?

Let $m \in \mathbb{N}$. Then $C_0^m(\mathbb{R}^n)$ is the collection of all (complex-valued) $f \in C^m(\mathbb{R}^n)$ with compact support in $\mathbb{R}^n$.

Theorem 8. Let $n \in \mathbb{N}$, $m \in \mathbb{N}$ and $1 \le p < \infty$. For $t > 0$ let $w(t)$ Lebesgue measurable with $0 < w(t) < \infty$,

$$t^{n-1} w(t) \in L_1(0,a) \quad \text{for any } a > 0,$$

$$\int_1^\infty t^{-n/p} \Big(\sup_{\tau > 0} \frac{w(\tau)}{w(\tau t)} \Big)^{1/p} \frac{\mathrm{d}t}{t} < \infty.$$

Then there is an $c > 0$ such that for all $f \in C_0^m(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |f(x)|^p w(|x|) \, \mathrm{d}x \le c \int_{\mathbb{R}^n} |x|^{mp} \sum_{|\alpha| = m} |D^\alpha f(x)|^p \, w(|x|) \, \mathrm{d}x.$$

Remark 9. Direct proof, $c = c(n,m,p,w)$ explicit estimates (surely not sharp). Many papers and books about Hardy inequalities. Books by Kufner, Opic, Maligranda, Persson etc.
Applies in particular to all positive monotonically increasing w.

Let $B_\delta = \{x \in \mathbb{R}^n : \ |x| < \delta\}$ where $\delta > 0$. Let $m \in \mathbb{N}$. Then $C_0^m(B_\delta)$ collects all $f \in C^m(\mathbb{R}^n)$ with compact support in B_δ.

Theorem 10. Let $n \in \mathbb{N}$, $m \in \mathbb{N}$, $1 \le p < \infty$ and $\sigma \in \mathbb{R}$. For $t > 0$ let $w(t)$ Lebesgue measurable with $0 < w(t) < \infty$,

$$\max(1, -\log t)^{\sigma p} t^{n-1} w(t) \in L_1(0,a) \quad \text{for any } a > 0,$$

$$\int_1^\infty t^{-n/p} \Big(\sup_{\tau > 0} \frac{w(\tau)}{w(\tau t)} \Big)^{1/p} \frac{\mathrm{d}t}{t} < \infty.$$

Then for some $1 > \delta > 0$, $c > 0$ and all $f \in C_0^m(B_\delta)$,

$$\int_{\mathbb{R}^n} \big| \log |x| \, \big|^{\sigma p} \, |f(x)|^p w(|x|) \, \mathrm{d}x \le c \int_{\mathbb{R}^n} |x|^{mp} \big| \log |x| \, \big|^{\sigma p} \sum_{|\alpha| = m} |D^\alpha f(x)|^p w(|x|) \, \mathrm{d}x.$$

Proof: Reduction to Theorem 8 replacing $f(x)$ by $(-\log |x|)^\sigma f(x)$.

U_B unit ball of the (complex) quasi-Banach space B. Let $T \in L(B_1, B_2)$ linear compact operator mapping B_1 in B_2. Then

$$e_k(T) = \inf \left\{ \varepsilon > 0 : \; TU_{B_1} \subset \bigcup_{j=1}^{2^{k-1}} \left(b_j + \varepsilon U_{B_2} \right) \right\}$$

for some $b_1, \ldots, b_{2^{k-1}} \in B_2$, is the kth entropy number, $k \in \mathbb{N}$.

$$a_k(T) = \inf \left\{ \| T - S \| : \; S \in L(B_1, B_2), \; \text{rank } S < k \right\}$$

for $k \in \mathbb{N}$, rank $S = \dim(\text{image } S)$ is the kth approximation number.

H complex separable Hilbert space. A positive definite self-adjoint operator in H with pure point spectrum,

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots \to \infty \quad \text{if } k \to \infty.$$

Then

$$\mathrm{id} : \quad \mathrm{dom}\ A^{1/2} \hookrightarrow H \quad \text{is compact}$$

and

$$\lambda_k^{-1/2} = a_k \left(A^{-1/2} : H \hookrightarrow H \right) \sim a_k \big(\mathrm{id} : \; \mathrm{dom}\ A^{1/2} \hookrightarrow H \big).$$

B unit ball in $\mathbb{R}^n$. For $m \in \mathbb{N}_0$ and $\sigma \geq 0$ let

$$b_{m,\sigma}(x) = |x|^m \big(1 - \log|x|\big)^\sigma, \qquad x \in B.$$

For $1 \leq p < \infty$ and $f \in C_0^m(B)$ let

$$\|f\,|E^m_{p,\sigma}(B)\| = \Big(\int_B b^p_{m,\sigma}(x) \sum_{|\alpha|=m} |D^\alpha f(x)|^p \,\mathrm{d}x\Big)^{1/p}.$$

Proposition 11. Let $1 \leq p < \infty$, $m \in \mathbb{N}$ and $\sigma \geq 0$. There is a constant $c > 0$ such that for all $f \in C_0^m(B)$,

$$\|f\,|L_p(B)\| \leq c\,\|f\,|E^m_{p,\sigma}(B)\|.$$

Furthermore, $C_0^m(B)$ furnished with $\|\cdot\,|E^m_{p,\sigma}(B)\|$ is closable in $L_p(B)$.

Proof. Use the above Hardy inequalities.

Notation: $E^m_{p,\sigma}(B)$ is the completion of $C_0^m(B)$ in the above norm, weighted Sobolev spaces.

So far

$$\mathrm{id}: \; E^m_{p,\sigma}(B) \hookrightarrow L_p(B), \qquad m \in \mathbb{N}, \quad 1 \le p < \infty, \quad \sigma \ge 0.$$

Theorem 12. Let $m \in \mathbb{N}$, $1 \le p < \infty$, $\sigma \ge 0$.

(i) id compact if, and only if, $\sigma > 0$.

(ii) Let $\sigma > \frac{n+m}{n}$. Then

$$e_k(\mathrm{id}) \sim a_k(\mathrm{id}) \sim k^{-\frac{m}{n}}, \qquad k \in \mathbb{N}.$$

(iii) Let $0 < \sigma \le \frac{n+m}{n}$. Then there is a number $c > 0$ and for any $\varepsilon > 0$ a number $c_\varepsilon > 0$ such that

$$c\, k^{-\frac{m}{n}} \le e_k(\mathrm{id}) \le c_\varepsilon\, k^{-\sigma \frac{m}{n+m} + \varepsilon}, \qquad k \in \mathbb{N},$$

and

$$c\, k^{-\min(\sigma, \frac{m}{n})} \le a_k(\mathrm{id}) \le c_\varepsilon\, k^{-\sigma \frac{m}{n+m} + \varepsilon}, \qquad k \in \mathbb{N}.$$

Proof. Basic ideas: Decompositions based on the above Hardy inequalities similar as for entropy numbers in limiting embeddings.

$$\varphi(t) = 1 \text{ if } t \geq 1/2, \qquad \varphi(t) = 0 \text{ if } t \leq 1/4$$

on $\mathbb{R}$. Let $\varphi_0(x) = \varphi(|x|)$ in $\mathbb{R}^n$.

$$\varphi_j(x) = \varphi_0(2^j x) - \varphi_0(2^{j-1}x) \in D(\mathbb{R}^n), \qquad j \in \mathbb{N}.$$

Decomposition of $f = \sum_{j=0}^{\infty} \varphi_j f$. Application of the above Hardy inequalities reduces the problem to

$$\mathrm{id}^j : \quad E^m_{p,\sigma}(B) \hookrightarrow \overset{\circ}{W}{}^m_p(B^j), \qquad \mathrm{id}^j = \varphi_j f, \qquad j \in \mathbb{N},$$

where

$$B^j = \left\{ x \in \mathbb{R}^n : \; 2^{-j-2} < |x| < 2^{-j} \right\}$$

annuli around the origin. Then classical assertions for unweighted Sobolev spaces, based on

$$\|\mathrm{id}^j\| \leq c\, 2^{jm} j^{-\sigma}, \qquad j \in \mathbb{N}.$$

The assertions for entropy and approximation numbers related to id^j are clipped together in the same was as in Edmunds-T in connection with limiting embeddings for entropy and approximation numbers.

Now $H = L_2(B)$ underlying Hilbert space, B again unit ball in $\mathbb{R}^n$.

$$E^m_\sigma(B) = E^m_{2,\sigma}(B), \qquad m \in \mathbb{N}, \quad \sigma > 0,$$

based on quadratic form

$$E^m_\sigma(f,g) = \int_B b^2_{m,\sigma}(x) \sum_{|\alpha|=m} D^\alpha f \cdot D^\alpha \bar{g}(x)\, \mathrm{d}x,$$

$$\|f \,|E^m_\sigma(B)\| = E^m_\sigma(f,f)^{1/2},$$

with

$$b^2_{m,\sigma}(x) = |x|^{2m}\big(1 - \log|x|\big)^{2\sigma}, \qquad x \in B, \quad m \in \mathbb{N}, \quad \sigma > 0.$$

Closable quadratic form in $L_2(B)$. Generates self-adjoint positive definite operator A^m_σ,

$$A^m_\sigma f = (-1)^m \sum_{|\alpha|=m} D^\alpha \big(b^2_{m,\sigma} D^\alpha f\big), \qquad \mathrm{dom}\ \sqrt{A^m_\sigma} = E^m_\sigma(B),$$

with pure point spectrum.

A^m_σ as above, $\{\lambda_k(A^m_\sigma)\}$ eigenvalues,

$$0 < \lambda_1(A^m_\sigma) \le \lambda_2(A^m_\sigma) \le \cdots \le \lambda_k(A^m_\sigma) \le \cdots \to \infty \quad \text{if } k \to \infty.$$

As before,

$$\lambda_k^{-1/2}(A^m_\sigma) \sim a_k\big(\mathrm{id}: \ E^m_\sigma(B) \hookrightarrow L_2(B)\big), \qquad k \in \mathbb{N}.$$

Theorem 13. (i) If $\sigma > \frac{n+m}{n}$ then

$$\lambda_k(A^m_\sigma) \sim k^{\frac{2m}{n}}, \qquad k \in \mathbb{N}.$$

(ii) If $0 < \sigma \le \frac{n+m}{n}$ then there is a number $c > 0$ and for any $\varepsilon > 0$ a number $c_\varepsilon > 0$ with

$$c_\varepsilon\, k^{2\sigma \frac{m}{n+m} - \varepsilon} \le \lambda_k(A^m_\sigma) \le c\, k^{2\min(\sigma, \frac{m}{n})}, \qquad k \in \mathbb{N}.$$

Proof. Reduction to Theorem 12.

Remark 14. Breaking point for σ? First candidate $\sigma = m/n$.

Remark 15. Main ingredients: Hardy inequalities, decomposition techniques, weights $w(t) \sim 1$, $w(t) \sim t^\delta$, $w(t) \sim t^\delta |\log t|^\sigma$. But works for more general $w(t)$ and more sophisticated assertions for the distribution of eigenvalues can be expected.