Fedor Stonyakin (Simferopol, Ukraine)

COMPACT SUBDIFFERENTIAL AND ITS APPLICATION TO GENERALIZED RADON-NIKODYM PROPERTY IN FRECHET SPACES

Introduction.

Lebesgue thm.: $F : I = [a;b] \to \mathbb{R}$ is AC $\Rightarrow$ F is differ. a.e. and

$$F(x) = F(a) + (L)\int_a^x F'(t)dt \qquad \forall x \in [a;b].$$

Indefinite Bochner integral — $F : [a;b] \to E$, **E is a LCS:**

$$F(x) = F(a) + (B)\int_a^x f(t)dt \quad \forall x \in [a;b] \qquad (F \in W_1^1(I,E)).$$

(i) Absolute continuity: $F \in AC(I,E)$.

(ii) Differentiability a.e. (E is Banach sp.).

$\exists F \in AC(I,E),\ F$ is not indef. Bocher integral.

Radon-Nikodym prop. (RNP): $F : I = [a;b] \to E$ is AC $\Rightarrow$

$$F(x) = F(a) + (B)\int_a^x f(t)dt \quad \forall x \in [a;b].$$

Spaces with (RNP): reflexive spaces, ℓ_1.

Spaces without (RNP): ℓ_∞, c, c_0, $L_1([a;b])$, $L_\infty([a;b])$ and $C[a;b]$.

Main problem: How we can operate in spaces without the Radon-Nikodym property?

Our approach: AC $\longmapsto$ AC in some E_C (AC_K):

$\|\cdot\|_E \ \longmapsto\ \|\cdot\|_C$, C **is an a. c. compact set in** E, where

$$\|x\|_C = \inf\{\lambda \in \mathbb{R} \ \mid\ x \in \lambda \cdot C\}.$$

1. Compact subdifferential as apparatus for investigation of mappings from AC_K

Def. 1. (I. V. Orlov) $F : I = [a;b] \longrightarrow E$, $x \in I$, $\delta > 0$. *Partial convex subdifferential* of F:

$$\partial_K F(x,\delta) = \overline{co}\left\{ \frac{F(x+h)-F(x)}{h} \;\mid\; 0 <\mid h \mid< \delta \right\}.$$

$F : I \longrightarrow E$ is *compact subdifferentiable* at $x \in I$ if

$$\exists\, \partial_K F(x) = K\text{-}\lim_{\delta \to +0} \partial_K F(x,\delta).$$

Thm. 1. *(General properties of $\partial_K F$).*

(i) $\partial_K(\lambda \cdot F)(x) = \lambda \cdot \partial_K F(x)$;

(ii) $\partial_K(F_1 + F_2)(x) \subset \partial_K F_1(x) + \partial_K F_2(x)$;

(iii) $A \in L(E_1; E_2) \Rightarrow (\partial_K(F \circ A)(x) = A(\partial_K F(x)))$.

Thm. 2. $F \in AC_K(I,E) \Longrightarrow \exists\ \partial_K F$ *a.e. on I.*

Thm. 3. *If* $F \in C[a;b]$ *and compact subdifferentiable on* $[a;b]\backslash e$, *where* $mes(e) = mes^w F(e) = 0$ *then*

$$\frac{F(b)-F(a)}{b-a} \in \overline{co}\ \partial_K F([a;b]\backslash e)\ . \tag{1}$$

Some other subdifferentials:

1. Subdifferentials for convex and concave functions.
2. Clarke and Michel-Penot subdifferentials.
3. Frechet subdifferentials.

2. Counterpart of Denjoy-Young-Saks's theorem and its application to Bochner integral

Thm. 4. *(Corol. from Denjoy-Young-Saks's theorem)* $f : \mathbb{R} \to \mathbb{R}$. *For a.e.* $x \in [a; b]$ *or* f *is differentiable at* x, *or* $\exists$ *an infinite derivative number of* f *at* x.

Let $F : [a; b] \to E$, where E is a Frechet space.

Thm. 5. *Let* $F : [a; b] \to E$ *be a separable-valued a.e. on* $[a; b]$. *Then one from the following conditions is true a.e. on* $I = [a; b]$*:*

(i) $\exists F'(x)$,

(ii) $\exists$ *derivative number* $\widehat{\partial} F(x) = \infty$,

(iii) any sequence $\frac{\triangle F(x,h_k)}{h_k}$ $(h_k \to 0)$ *doesn't have limit point.*

Application to Bochner integral

Thm. 6. $F \in AC(I,E)$ *and compact subdiffer. a.e. on* $I \Longrightarrow$
$\Longrightarrow \exists\ f : I \to E$ *is Bochner integr. on* I *and*

$$F(x) = F(a) + (B)\int_a^x f(t)dt \quad (a \leqslant x \leqslant b)\ .$$

Thm. 7. $F \in AC_K(I,E) \Longrightarrow \exists\ f : I \to E$*:*

$$F(x) = F(a) + (B)\int_a^x f(t)dt,\ x \in [a;b].$$

Thm. 8. $F \in AC_K(I,E) \Longleftrightarrow$

(i) $F(x) = F(a) + (B)\int_a^x f(t)dt,\ x \in [a;b] \quad (F \in W_1^1(I,E))$;

(ii) $\int_a^b \|f(t)\|_C dt < \infty$ *for some* $C \in \mathcal{C}(E)$.

3. Compact form of (RNP)

$$AC_K(I,E) \subset W_1^1(I,E) \subset AC(I,E). \tag{2}$$

Def. 2. $E \in (RNP)_K$: $AC_K(I,E) = W_1^1(I,E)$.

Thm. 9. Each Frechet space $E \in (RNP)_K$.

Rem. 1. There exist non-Frechet LCS, such that:

$$\begin{array}{ll} E \in (RNP) \bigcap (RNP)_K, & E \in (RNP),\ E \notin (RNP)_K \\ E \notin (RNP),\ E \in (RNP)_K, & E \notin (RNP),\ E \notin (RNP)_K. \end{array}$$

Thm. 10. E — Frechet space $\Longrightarrow$

$$W_1^1(I,E) = \bigcup_{C \in \mathcal{C}(E)} AC(I,E_C) = \bigcup_{C' \in \mathcal{C}(E)} W_1^1(I,E_{C'}). \tag{3}$$

4. Limit form of (RNP)

Spaces under consideration. $(E, \{\|\cdot\|_n\}_{n=1}^{\infty})$ — Frechet space.

(i) $W_1^1(I,E) = \left\{ F(x) = F(a) + (B)\int_a^b F'(t)dt \right\},$

$$\|F\|^n = \|F(a)\|_n + \int_a^b \|F'(t)\|_n dt.$$

(ii) $AC(I, E_C)$: $\|F\|^C = \|F(a)\|_C + \int_a^b \|F'(t)\|_C dt.\quad (C \in \mathcal{C}(E))$

(iii) $W_1^1(I, E_{C'})$: $\|F\|^{C'} = \|F(a)\|_{C'} + \int_a^b \|F'(t)\|_{C'} dt.\quad (C' \in \mathcal{C}(E))$

Thm. 11. *For each Frechet space E* **the limit form of (RNP) holds**: $W_1^1(I,E) \overset{top}{=} \varinjlim_{C\in\mathcal{C}(E)} AC(I, E_C) \overset{top}{=} \varinjlim_{C'\in\mathcal{C}(E)} W_1^1(I, E_{C'}).$

5. Applications

Corol. 1. *E is a Frechet sp., X is a LCS, $A : W_1^1(I, E) \to X$ is a linear op. Equivalence:*

(i) *A is continuous on $W_1^1(I, E)$;*

(ii) *A is continuous on each $W_1^1(I, E_C)$, $C \in \mathcal{C}(E)$;*

(iii) *A is continuous on each $AC(I, E_{C'})$, $C' \in \mathcal{C}(E)$.*

Corol. 2. *If $F : [a; b] \to E$ is cont. on $[a; b]$ and differ. on $[a; b]\backslash e$, $mes(e) = 0$, $mes^w F(e) = 0$ and $F'([a; b]\backslash e)$ is bounded $\Longrightarrow$*

$$\frac{F(b) - F(a)}{b - a} \in \overline{co}_{E_C} \; F'\left([a; b]\backslash e\right). \tag{4}$$

6. Some open problems:

1. An exact description of LCS with $(RNP)_K$.
2. Limit form of (RNP) in LCS.
3. Other infinite-dimensional integrals.

Other subdifferentials

Clarke and Michel-Penot subdifferentials

$$f^{\uparrow}_{Cl}(x_0;g) = \limsup_{x'\to x_0,\ \alpha\downarrow 0} \frac{1}{\alpha}\left[f(x'+\alpha g) - f(x')\right] \ ;$$

$$f^{\uparrow}_{mp}(x_0;g) = \sup_{q\in\mathbb{R}} \left\{ \limsup_{\alpha\downarrow 0} \frac{1}{\alpha}\left[f(x_0+\alpha(g+q)) - f(x_0+\alpha q)\right] \right\} \ .$$

$$f^{\uparrow}_{Cl}(x_0;g) = \max_{\ell\in\partial_{Cl}f(x_0)} (\ell\cdot g) \ ; \quad f^{\uparrow}_{mp}(x_0;g) = \max_{\ell\in\partial_{mp}f(x_0)} (\ell\cdot g) \ .$$

Frechet subdifferentials

$$\partial^-_F f(x_0) = \{ v\in\mathbb{R} \ ; \ \exists\ g\in C^1(U(x_0)) : v = g'(x_0)\} \ ,$$

$$\text{and } (f-g)(x_0) = \min_{x\in U(x_0)} (f-g)(x)$$

$$\partial^+_F f(x_0) = \{ v\in\mathbb{R} \ ; \ \exists\ g\in C^1(U(x_0)) : v = g'(x_0)\} \ ,$$

$$\text{and } (f-g)(x_0) = \max_{x\in U(x_0)} (f-g)(x)$$

Ex. 1. $\mathcal{L}_1(I=[0;1])$ be a space of real-valued meas. funct. $x=\xi(t)$ endowed with the norm $\|x\| = \int\limits_0^1 |\, \xi(t) \,|\, dt$; $y(\cdot) : I \to \mathcal{L}_1(I)$:

$$y(s)(t) = \begin{cases} 1, \text{ for } & 0 \leqslant t \leqslant s \ ; \\ 0, \text{ for } & s < t \leqslant 1 \ . \end{cases}$$

Ex. 2. $T=[0;1]$, $E=\{\varphi : T \to \mathbb{R}\}$ with a pointwise conv.,
$F : [0;1] \to E$:

$$F(s,\theta) = \begin{cases} \int\limits_\theta^s \frac{d\tau}{\sqrt{\tau-\theta}}, & (0 \leqslant s \leqslant 1,\ 0 \leqslant \theta \leqslant s)\,, \\ 0, & (0 \leqslant s \leqslant 1,\ s < \theta \leqslant 1)\,. \end{cases}$$

$$\exists f : T \to E : \quad F(s) = (B)\int\limits_0^s f(u)du \quad (0 \leqslant s \leqslant 1)\ .$$

F has infinite «deriv. numb.» $\forall s \in [0;1] \Longrightarrow F \notin AC_K(I,E)$.