Some new applications of asymptotic behaviour of entropy and Weyl numbers to spectral theory

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Eigenvalues of compact operators

Let $B : X \to X$ compact, linear with eigenvalue sequence $\{\mu_k(B)\}_{k\in\mathbb{N}}$, counted with respect to their algebraic multiplicity and ordered by decreasing modulus. If B has only finitely many distinct eigenvalues different from zero, we put $\mu_m(B) = 0$ for k sufficiently large.

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3. Negative spectrum of Schödinger type operators $H_\gamma = (\mathrm{id} - \Delta)^{\varkappa/2} - \gamma V(x)$, $V(x) \geq 0$, $\gamma > 0$.

$$\#\{\sigma_p(H_\gamma) \cap (-\infty, 0]\} \quad \text{as} \quad \gamma \to \infty$$

The Birman-Schwinger principle relates the behaviour of $\#\{\sigma_p(H_\gamma) \cap (-\infty, 0]\}$ to the behaviour of $\mu_m(B)$ for compact operator $B = \sqrt{V}(\mathrm{id} - \Delta)^{-\varkappa/2}\sqrt{V}$.

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Some of the presented estimates can be improved using the Weyl numbers of Sobolev embeddings (in particular if one works with weithed function spaces with logarithmic type weights) (joint work with Alicja Gąsiorowska)

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Mutiplicativity property - $s_{k+m-1}(BA) \le s_k(B)s_m(A)$; $s_1(B) = \|B\|$, where $s_k = e_k$ or $s_k = x_k$

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Problem: How to calculate asymptotic behaviour of $e_k(B)$, $x_k(B)$ if $k \to \infty$? Hint: to factorize B though compact Sobolev embeddings and to calculate e_k or x_k for the embeddings.

Function spaces - definitions

1. Besov spaces, $s \in \mathbb{R}$ and $0 < p, q \leq \infty$,

$$B^s_{p,q}(\mathbb{R}^n) = \{f \in S' : \|f|B^s_{p,q}\| = \Big(\sum_{j=0}^{\infty} 2^{sqj} \|\mathcal{F}^{-1}\varphi_j \mathcal{F} f\|_p^q\Big)^{1/q} < \infty\}.$$

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2. Besov spaces on domains, $\Omega \subset \mathbb{R}^n$ - an open set $\Omega \neq \mathbb{R}^n$, $p, q > 1$.

$$\bar{B}^s_{p,q}(\Omega) = \begin{cases} \{f : f = g|_\Omega,\ g \in B^s_{p,q}(\mathbb{R}^n)\}, & \text{if} \quad s \leq 0, \\ \{f : f = g|_\Omega,\ g \in B^s_{p,q}(\mathbb{R}^n)\ \mathrm{supp} g \subset \overline{\Omega}\} & \text{if} \quad s > 0, \end{cases}$$

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3. Weighted Besov spaces with a Muckenhaupt weight w.

$$L_p(\mathbb{R}^n, w) := \{f : \int |f(x)|^p w(x) dx < \infty\},$$

$$\|f|B^s_{p,q}(\mathbb{R}^n, w)\| = \Big(\sum_{j=0}^{\infty} 2^{sqj} \|\mathcal{F}^{-1}\varphi_j \mathcal{F} f|L_p(\mathbb{R}^n, w)\|^q\Big)^{1/q}$$

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Theorem

If Ω is an unbounded domain with finite Lebesgue measure then the embedding (1) is compact if and only if (2) holds and the corresponding entropy numbers satisfy the estimates (3).

Entropy numbers of Sobolev embedding on domain

- We recall that an unbounded domain Ω in $\mathbb{R}^n$ is called quasi-bounded if

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Examples

Let $\alpha > 0$. The open sets $\omega_\alpha, \Omega_\alpha \subset \mathbb{R}^2$

$$\begin{aligned} \omega_\alpha &= \{(x,y)\in\mathbb{R}^2 : \; |y| < x^{-\alpha}, x > 1\} \qquad \text{and} \\ \Omega_\alpha &= \{(x,y)\in\mathbb{R}^2 : \; |y| < |x|^{-\alpha}\} \quad \text{are quasi-bounded.} \end{aligned}$$

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Theorem

If Ω is not quasi-bounded then the embedding (1) is never compact.

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$$b(\Omega) = \sup \Big\{ t \in \mathbb{R}_+ : \limsup_{j\to\infty} b_j(\Omega) 2^{-jt} = \infty \Big\}. \tag{4}$$

- For any nonempty open set $\Omega \subset \mathbb{R}^n$ we have $n \le b(\Omega) \le \infty$. If Ω is of finite measure, then $b(\Omega) = n$. If Ω is unbounded and not quasi-bounded, then $b(\Omega) = \infty$.

Box packing number of an open set - examples

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Let $\alpha > 0$, $\omega_\alpha, \Omega_\alpha \subset \mathbb{R}^2$ be as above. Then

$$b(\omega_\alpha) = \begin{cases} \frac{1}{\alpha} + 1 & \text{if } 0 < \alpha < 1, \\ 2 & \text{if } \alpha \geq 1, \end{cases} \qquad b(\Omega_\alpha) = \begin{cases} \frac{1}{\alpha} + 1 & \text{if } 0 < \alpha < 1, \\ \alpha + 1 & \text{if } \alpha \geq 1. \end{cases}$$

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There are quasi-bounded domains such that $b(\Omega) = \infty$.

Quasi-bounded domains - compactness of embeddings

Theorem

(i) *Let $b(\Omega) < \infty$. The embedding*

$$\bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega) \tag{5}$$

is compact if

$$\delta := s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} > \frac{b(\Omega)}{p^*} = b(\Omega)\Big(\frac{1}{p_2} - \frac{1}{p_1}\Big)_+ . \tag{6}$$

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If the embedding (5) is compact and $\frac{1}{p^*} = 0$, then $\delta > 0$.
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If the embedding (5) is compact and $\frac{1}{p^*} > 0$, *then* $\delta \geq \frac{b(\Omega)}{p^*}$.
(ii) *If* $b(\Omega) = \infty$, *then the embedding (5) is compact if, and only if,* $p_1 \leq p_2$ *and*

$$s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} > 0 . \tag{7}$$

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$$0 < \liminf_{j\to\infty} b_j(\Omega) 2^{-jb(\Omega)} \le \limsup_{j\to\infty} b_j(\Omega) 2^{-jb(\Omega)} < \infty \;, \tag{8}$$

then

$$e_k\left(\bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega)\right) \sim k^{-\gamma} \tag{9}$$

with

$$\gamma = \frac{s_1 - s_2}{b(\Omega)} + \frac{b(\Omega) - n}{b(\Omega)}\left(\frac{1}{p_1} - \frac{1}{p_2}\right).$$

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$$\textit{with} \qquad \gamma = \frac{s_1 - s_2}{b(\Omega)} + \frac{b(\Omega) - n}{b(\Omega)}\left(\frac{1}{p_1} - \frac{1}{p_2}\right).$$

Corollary

Let Ω *be of finite Lebesgue measure. If the embedding is compact, then* $\gamma = \frac{s_1 - s_2}{n}$.

Quasi-bounded domains - inverse entropy problem

What is the possible asymptotic behaviour of entropy numbers of the compact embedding of the function spaces defined on domains?

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Theorem

Let $s_1, s_2 \in \mathbb{R}$, $0 < p_1, p_2 \leq \infty$ *and* $0 < q_1, q_2 \leq \infty$. *We assume that* $\frac{s_1 - s_2}{n} > \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+$.
For positive real γ, *such that* $\frac{s_1 - s_2}{n} \geq \gamma > \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+$, *there exists a quasi-bounded domain* Ω *in* $\mathbb{R}^n$ *such that*

$$e_k\left(\bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega)\right) \sim k^{-\gamma}, \quad k \in \mathbb{N}\,. \tag{10}$$

If (10) holds for some quasi-bounded domain Ω *in* $\mathbb{R}^n$ *and* $b(\Omega) < \infty$, *then* $\frac{s_1 - s_2}{n} \geq \gamma > \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+$.

Elliptic operators on quasi-bounded domains

$$\text{Let} \qquad A(x,D) = \sum_{|\alpha|\le 2m} a_\alpha(x)\partial^\alpha$$

be a formally self-adjoint, uniformly strongly elliptic differential operator of order $2m$, $m \in \mathbb{N}$, with real valued coefficients $a_\alpha \in C^\infty(\Omega)$ which are uniformly bounded and uniformly continuous for $|\alpha| \le 2m$. We assume that A is a positive self-adjoint operator in $L_2(\Omega)$. Then $A = A(x,D)$ is an operator with discrete spectrum $\sigma(A)$ of eigenvalues having no finite accumulation point.

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Theorem

Let Ω be quasi-bounded domain in $\mathbb{R}^n$, such that $b(\Omega) < \infty$ and (8) holds. Let $\lambda_1, \lambda_2, \dots$ be eigenvalues of A ordered by their magnitude and counted according to their multiplicities. Then

$$\lambda_k \sim k^{\frac{2m}{b(\Omega)}}, \qquad k \in \mathbb{N}\,.$$

Elliptic operators on quasi-bounded domains-examples

Examples

Let $\alpha > 0$ and $\alpha \neq 1$. For the open set $\Omega_\alpha \subset \mathbb{R}^2$ we have the following formula for the eigenvalues of the Dirichlet Laplacian

$$\lambda_k(-\Delta) \sim \begin{cases} k^{\frac{2\alpha}{1+\alpha}} & \text{if} \quad 0 < \alpha < 1\,, \\ k^{\frac{2}{1+\alpha}} & \text{if} \quad \alpha > 1\,. \end{cases} \tag{11}$$

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The assumption (8) is sufficient but not necessary to get the estimates of corresponding entropy numbers. For the domain Ω_α with $\alpha = 1$ one gets

$$e_k\left(\bar{B}^{s_1}_{p_1,q_1}(\Omega_1) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega_1)\right) \sim k^{-\frac{s_1-s_2}{2}}(\log k)^{\frac{s_1-s_2}{2}-(\frac{1}{p_1}-\frac{1}{p_2})},$$

$$\text{and} \qquad \lambda_k(-\Delta) \sim k \log k\,.$$

Muckenhaupt weights and elliptic operators on $\mathbb{R}^n$

We regards the following weights

$$w_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(x) = \begin{cases} |x|^{\alpha_1}(1-\log|x|)^{\alpha_2}, & \text{if} \quad |x| \le 1\,, \\ |x|^{\beta_1}(1+\log|x|)^{\beta_2}, & \text{if} \quad |x| > 1\,, \end{cases} \tag{12}$$

$$\boldsymbol{\alpha} = (\alpha_1,\alpha_2),\ \alpha_1 > -n,\ \alpha_2 \in \mathbb{R}, \quad \boldsymbol{\beta} = (\beta_1,\beta_2),\ \beta_1 > -n,\ \beta_2 \in \mathbb{R}. \tag{13}$$

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This covers weights of purely polynomial growth both near 0 and ∞,

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Problem: Criteria for compactness and entropy numbers of embeddings of type

$$\mathrm{id} : B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n),$$

where $s_2 \le s_1$, $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 \le \infty$,

Embeddings with Muckenhaupt weights - motivations

- To estimate the negative spectrum $\#\{\sigma_p(H_\gamma) \cap (-\infty, 0]\}$ of operators of type

$$H_\gamma = A - \gamma V(x) \quad \text{as} \quad \gamma \to \infty \,,$$

where A is an elliptic pseudodifferential operator of order $\varkappa > 0$, positive-definite and self-adjoint in $L_2(\mathbb{R}^n)$, e.g. $A = (\mathrm{id} \, - \Delta)^{\varkappa/2}$, and $V(x)$ is a positive, real (singular) function.

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- The Birman-Schwinger principle. Let $B = \sqrt{V} A^{-1} \sqrt{V}$ be compact.

$$\sigma_e(H_\gamma) = \sigma_e(A) \quad \text{and} \quad \#\{\sigma(H_\gamma) \cap (-\infty, 0]\} = \#\{\sigma_p(H_\gamma) \cap (-\infty, 0]\} \\ = \#\{k \in \mathbb{N} : \; \mu_k(B) \geq 1\} \; < \; \infty \,.$$

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$$= \#\{k \in \mathbb{N} : \mu_k(B) \geq 1\} < \infty \, .$$
- By factorization we can reduce the estimates for B to the estimates for the Sobolev embeddings of weighted spaces

Negative spectrum - purely polynomial estimates for V.

Theorem

Let $\varkappa > 0$, $w_{\alpha,\beta} V \in L_\infty(\mathbb{R}^n)$, *and*

$$-n < \alpha < n, \quad 0 < \beta < n, \quad \varkappa > \alpha_+ , \quad \varkappa \neq \beta. \tag{15}$$

Then

$$\#\{\sigma_p(H_\gamma) \cap (-\infty, 0]\} \leq c \left(\gamma \| w_{\alpha,\beta} V | L_\infty \|^2\right)^{\frac{n}{\min(\varkappa,\beta)}} , \quad \gamma \to \infty. \tag{16}$$

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Examples

Let $\quad H_\gamma = A - \gamma |x|^{-\mu} , \qquad 0 < \mu < n, \quad \varkappa > \mu, \quad$ i.e. $\quad V(x) = |x|^{-\mu}.$

Then $\quad \#\{\sigma_p(H_\gamma) \cap (-\infty, 0]\} \leq C \gamma^{\frac{n}{\mu}}, \qquad \gamma \to \infty.$

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Limiting cases: $\varkappa = \beta > 0$ or $\varkappa = \alpha > 0$?

Negative spectrum - limiting cases

Theorem

Let $0 < \varkappa < n$ *and* $w_{(\alpha,\beta)} V \in L_\infty(\mathbb{R}^n)$.

Negative spectrum - limiting cases

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Let $0 < \varkappa < n$ *and* $w_{(\boldsymbol{\alpha},\boldsymbol{\beta})} V \in L_\infty(\mathbb{R}^n)$.

(i) *Let* $\alpha_1 = \varkappa$, $\alpha_2 > 2\frac{\varkappa}{n}$, $0 < \beta_1 = \beta < n$ *and* $\beta_2 = 0$. *Then there exists a positive constant* $C > 0$ *independent of* γ, $\boldsymbol{\beta}$ *and* V *such that*

$$\#\{\sigma_p(H_\gamma) \cap (-\infty, 0]\} \leq C \begin{cases} \gamma^{\frac{n}{\varkappa}} & \text{if} \quad \varkappa < \beta\,, \\ \gamma^{\frac{n}{\beta}} & \text{if} \quad \varkappa > \beta\,, \\ \gamma^{\frac{n}{\varkappa}} \log \gamma & \text{if} \quad \varkappa = \beta, \end{cases} \qquad \gamma \to \infty.$$

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(ii) *Assume* $\max(0, \alpha_1) < \varkappa$ *and* $\beta_1 = \varkappa$. *Then there exists a positive constant* $C > 0$ *independent of* γ, $\boldsymbol{\beta}$ *and* V *such that*

$$\#\{\sigma_p(H_\gamma) \cap (-\infty, 0]\} \le C \begin{cases} \gamma^{\frac{n}{\varkappa}} & \text{if} \quad \frac{\varkappa}{n} < \beta_2, \\ \gamma^{\frac{n}{\varkappa}} (\log \gamma)^{1-\beta_2 \frac{n}{\varkappa}} & \text{if} \quad 0 \ge \beta_2, \end{cases} \qquad \gamma \to \infty.$$

Negative spectrum - limiting cases -examples

Examples

(i) Let $0 < \varkappa < n$, $\varepsilon > 0$, and

$$V(x) = \begin{cases} |x|^{-\varkappa}\big(1 - \log|x|\big)^{-2\frac{\varkappa}{n} - \varepsilon} & \text{if} \quad |x| < 1, \\ |x|^{-\varkappa}\big(1 + \log|x|\big)^{-\beta} & \text{if} \quad |x| \geq 1. \end{cases}$$

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Negative spectrum - limiting cases -examples

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(ii) Let $V(x) = \begin{cases} \big(1 - \log|x|\big)^{\alpha} & \text{if} \quad |x| < 1, \\ \big(1 + \log|x|\big)^{-\alpha} & \text{if} \quad |x| \geq 1, \end{cases}, \quad \alpha > 0 \quad \varkappa > 0.$

Then $\#\{\sigma_p(H_\gamma) \cap (-\infty, 0]\} \;\leq\; C \exp\big(\gamma^{\frac{1}{\alpha}}\big), \quad \gamma \to \infty.$

Logarithmic type weights

Let us regard the operator $B = b_2 A^{-1} b_1$ with

$$b_1 w_{(\mathbf{0},\boldsymbol{\beta})} \in L_{r_1}(\mathbb{R}^n), \quad b_2 w_{(\mathbf{0},\boldsymbol{\eta})} \in L_{r_2}(\mathbb{R}^n) \quad \text{where} \quad \beta_1 = \eta_1 = 0,$$

$$1 \le r_1' < p < r_2 \le \infty, \quad 1 < p < \infty, \quad \text{and} \quad \varkappa > n\left(\frac{1}{r_1} + \frac{1}{r_2}\right).$$

If $\beta_2 + \eta_2 > 0$ then the operator B is compact in $L_p(\mathbb{R}^n)$ and

$$|\mu_k(B)| \le c\|b_1 w_{(\mathbf{0},\boldsymbol{\beta})}|L_{r_1}(\mathbb{R}^n)\| \, \|b_2 w_{(\mathbf{0},\boldsymbol{\eta})}|L_{r_2}(\mathbb{R}^n)\| \times \begin{cases} k^{-\beta_2-\eta_2} & \text{if} \quad \beta_2 + \eta_2 \le \frac{1}{r_1} + \frac{1}{r_2} \\ k^{-\frac{1}{r_1}-\frac{1}{r_2}}(1+\log k)^{-\beta_2-\eta_2+\frac{1}{r_1}+\frac{1}{r_2}} & \text{if} \quad \beta_2 + \eta_2 > \frac{1}{r_1} + \frac{1}{r_2}. \end{cases} \tag{17}$$

Logarithmic type weights - entropy numbers

The last statement can be reduced by the Hölder inequality and the Carl inequality to the estimates of entropy numbers of Sobolev embeddings

$$\mathrm{id} : B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_{(\mathbf{0},\beta)}) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n), \quad \text{with} \quad \beta_1 = 0 \quad \text{and}$$

$$\frac{1}{p_1} = \frac{1}{p} + \frac{1}{r_1}, \quad \frac{1}{p} = \frac{1}{p_2} + \frac{1}{r_2}.$$

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In case of $\delta = s_1 - \frac{1}{p_1} - s_2 + \frac{1}{p_2} > 0$, $\beta_2 > 0$, then for all $k \in \mathbb{N}$,

$$e_k(\mathrm{id}) \sim \begin{cases} k^{-\frac{\beta_2}{p_1}}, & \text{if } \frac{\beta_2}{p_1} \le \frac{1}{p_1} - \frac{1}{p_2}, \\ k^{-\frac{1}{p_1}+\frac{1}{p_2}}(1+\log k)^{-\frac{\beta_2}{p_1}+\frac{1}{p_1}-\frac{1}{p_2}}, & \text{if } \frac{\beta_2}{p_1} > \frac{1}{p_1} - \frac{1}{p_2}. \end{cases}$$

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Alternative strategy: $x_k(\mathrm{id}\,)\sim ?$ plus the Pietsch inequality.

Logarithmic type weights - Weyl numbers

Theorem

Let $0 < p_1 \le p_2 \le \infty$, $1 < q_1, q_2 \le \infty$ *and* $s_1 - s_2 - d(\frac{1}{p_1} - \frac{1}{p_2}) > 0$. *Then*

$$x_k(\mathrm{id}) \sim (1 + \log k)^{-\frac{\beta_2}{p_1}} \begin{cases} k^{-(\frac{1}{p_1} - \frac{1}{p_2})} & \text{if} \quad 1 \le p_1 \le p_2 \le 2, \\ k^{-(\frac{1}{p_1} - \frac{1}{2})} & \text{if} \quad 1 \le p_1 < 2 < p_2 \le \infty, \\ 1 & \text{if} \quad 2 \le p_1 \le p_2 \le \infty. \end{cases}$$

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Remark

One can show also that

$$a_k(\mathrm{id}\,) \sim c_k(\mathrm{id}\,) \sim d_k(\mathrm{id}\,) \sim (1 + \log k)^{-\frac{\beta_2}{p_1}}.$$

Logarithmic type weights

Let us regard once more the operator $B = b_2 A^{-1} b_1$ with

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If $\beta_2 + \eta_2 > 0$ then the operator B is compact in $L_p(\mathbb{R}^n)$ and

$$|\mu_k(B)| \le \; c\|b_1 w_{(\mathbf{0},\boldsymbol{\beta})}|L_{r_1}(\mathbb{R}^n)\| \; \|b_2 w_{(\mathbf{0},\boldsymbol{\eta})}|L_{r_2}(\mathbb{R}^n)\| \times \tag{18}$$

$$\begin{cases} (1+\log k)^{-\beta_2-\eta_2} k^{-\frac{1}{r_1}-\frac{1}{r_2}} & \text{if} \quad \frac{1}{2} + \frac{1}{r_2} \le \frac{1}{p}, \\ (1+\log k)^{-\beta_2-\eta_2} k^{-\frac{1}{p}-\frac{1}{r_1}+\frac{1}{2}} & \text{if} \quad \frac{1}{2} - \frac{1}{r_1} < \frac{1}{p} < \frac{1}{2} + \frac{1}{r_2} \\ & \text{and} \quad \beta_2 + \eta_2 \le \frac{1}{p} + \frac{1}{r_1} - \frac{1}{2}, \end{cases}$$