Some new applications of asymptotic behaviour of entropy and Weyl numbers to spectral theory

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Eigenvalues of compact operators

Let $B : X \to X$ compact, linear with eigenvalue sequence $\{\mu_k(B)\}_{k \in \mathbb{N}}$, counted with respect to their algebraic multiplicity and ordered by decreasing modulus. If $B$ has only finitely many distinct eigenvalues different from zero, we put $\mu_m(B) = 0$ for $k$ sufficiently large.
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1. It is known that $\mu_k(B) \rightarrow 0$ if $k \rightarrow \infty$. What is the asymptotic behaviour of $\mu_k(B)$ if $k \rightarrow \infty$?

2. Asymptotic behaviour of eigenvalues of Dirichlet Laplacian on domains. If $\lambda_k$ eigenvalues of $-\Delta$ then $\lambda_k = \mu_k^{-1}(B)$ with compact $B = (-\Delta)^{-1}$ if $-\Delta$ has compact resolvent.
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3. Negative spectrum of Schrödinger type operators

$$H_\gamma = (\text{id} - \Delta)^{\kappa/2} - \gamma V(x), \ V(x) \geq 0, \ \gamma > 0.$$  

$$\#\{\sigma_p(H_\gamma) \cap (-\infty, 0]\} \quad \text{as} \quad \gamma \rightarrow \infty$$

The Birman-Schwinger principle relates the behaviour of

$$\#\{\sigma_p(H_\gamma) \cap (-\infty, 0]\}$$

to the behaviour of $\mu_m(B)$ for compact operator $B = \sqrt{V}(\text{id} - \Delta)^{-\kappa/2}\sqrt{V}$. 
Outline

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2 Elliptic operators on quasibounded domains
   (joint work with Hans-Gerd Leopold)
   *Compactness of embeddings of function spaces on quasi-bounded domains and the distribution of eigenvalues of related elliptic operators* (manuscript)
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Some of the presented estimates can be improved using the Weyl numbers of Sobolev embeddings (in particular if one works with weithed function spaces with logarithmic type weights) (joint work with Alicja Gąsiorowska)
Quantitative characterizations of compactness

Let $B \in L(X, Y)$ and $B_X$ be the closed unit ball in $X$. 
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**Entropy numbers.** The $k$-th entropy number of $B$, $k \in \mathbb{N}$, is

$$e_k(B) = \inf\{\varepsilon > 0 : B(B_X) \text{ can be covered by } 2^{k-1} \text{ balls of radius } \varepsilon \text{ in } Y\}.$$
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**Multiplicativity property** - $s_{k+m-1}(BA) \leq s_k(B)s_m(A)$; $s_1(B) = \| B \|$, where $s_k = e_k$ or $s_k = x_k$. 
Asymptotic behaviour of eigenvalues distributions

Let $B \in L(X, X)$ be a compact operator.
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$$\left( \prod_{j=1}^{k} |\mu_j(B)| \right)^{\frac{1}{k}} \leq \inf_{m \in \mathbb{N}} \frac{2^m}{2^{2k}} e_m(B) \quad \text{for all } k \in \mathbb{N},$$

where $\mu_j(B)$ are the eigenvalues of $B$. 
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especially,

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|\mu_k(B)| \leq \sqrt{2} e_k(B) \quad \text{for all } k \in \mathbb{N}.
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Pietsch’s Weyl-type inequality gives

$$\left( \prod_{j=1}^{2k} |\mu_j(B)| \right)^{\frac{1}{2k}} \leq 2\sqrt{2e} \left( \prod_{j=1}^{k} x_j(B) \right)^{\frac{1}{k}} \quad \text{for all } k \in \mathbb{N},$$
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Problem: How to calculate asymptotic behaviour of $e_k(B)$, $x_k(B)$ if $k \to \infty$? Hint: to factorize $B$ though compact Sobolev embeddings and to calculate $e_k$ or $x_k$ for the embeddings.
Besov spaces, $s \in \mathbb{R}$ and $0 < p, q \leq \infty$,

$$B^s_{p,q}(\mathbb{R}^n) = \{ f \in S' : \| f \|_{B^s_{p,q}} = \left( \sum_{j=0}^{\infty} 2^{sqj} \| \mathcal{F}^{-1} \varphi_j \mathcal{F} f \|_p \right)^{1/q} \}.$$
Function spaces - definitions

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2. **Besov spaces on domains**, \( \Omega \subset \mathbb{R}^n \) - an open set \( \Omega \neq \mathbb{R}^n \), \( p, q > 1 \).

\[
\bar{B}^s_{p,q}(\Omega) = \begin{cases} 
\{ f : f = g|_\Omega, \ g \in B^s_{p,q}(\mathbb{R}^n) \}, & \text{if } s \leq 0, \\
\{ f : f = g|_\Omega, \ g \in B^s_{p,q}(\mathbb{R}^n) \supp g \subset \overline{\Omega} \} & \text{if } s > 0,
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$$\| f \|_{B^{s}_{p,q}} = \inf \| g \|_{B^{s}_{p,q}}.$$  

3. Weighted Besov spaces with a Muckenhaust weight $w$.

$$L_p(\mathbb{R}^n, w) := \{ f : \int |f(x)|^p w(x)dx < \infty \},$$

$$\| f \|_{B^{s}_{p,q}(\mathbb{R}^n, w)} = \left( \sum_{j=0}^{\infty} 2^{sqj} \| \mathcal{F}^{-1} \varphi_j \mathcal{F} f \|_{L_p(\mathbb{R}^n, w)}^q \right)^{1/q}.$$
Entropy numbers of Sobolev embedding on domain

Let $\Omega$ be a bounded domain with sufficiently regular boundary (e.g. with Lipschitz boundary).
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The embedding $\bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega)$ (1) is compact if, and only if, $s_1 - s_2 - (\frac{n}{p_1} - \frac{n}{p_2})_+ > 0$. (2)
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Moreover $e_k(\bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega)) \sim k^{-\frac{s_1-s_2}{n}}$ (3)
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- Let $\Omega$ be an unbounded domain (with sufficiently regular boundary).
  What can we say about the properties of Sobolev embeddings?

  **Theorem**

  If $\Omega$ is an unbounded domain with finite Lebesgue measure then the embedding (1) is compact if and only if (2) holds and the corresponding entropy numbers satisfy the estimates (3).
We recall that an unbounded domain \( \Omega \) in \( \mathbb{R}^n \) is called quasi-bounded if

\[
\lim_{x \in \Omega, |x| \to \infty} \text{dist} (x, \partial \Omega) = 0.
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We recall that an unbounded domain $\Omega$ in $\mathbb{R}^n$ is called quasi-bounded if

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Examples

Let $\alpha > 0$. The open sets $\omega_\alpha, \Omega_\alpha \subset \mathbb{R}^2$

$$\omega_\alpha = \{(x, y) \in \mathbb{R}^2 : |y| < x^{-\alpha}, x > 1\} \quad \text{and}$$

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An unbounded domain is not quasi-bounded if, and only if, it contains infinitely many pairwise disjoint congruent balls.
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Elliptic operators on quasibounded domains

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Theorem

If $\Omega$ is not quasi-bounded then the embedding (1) is never compact.

L.Skrzypczak (UAM Poznań)
Box packing number of an open set

- The interesting case - quasi-bounded domains with infinite Lebesgue measure.
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b_j(\Omega) = \sup \left\{ k : \bigcup_{\ell=1}^{k} Q_{j,m_\ell} \subset \Omega , \quad Q_{j,m_\ell} \text{ being pairwise disjoint} \right\} , \quad j = 0, 1, \ldots
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$$b(\Omega) = \sup \left\{ t \in \mathbb{R}_+ : \limsup_{j \to \infty} b_j(\Omega) 2^{-jt} = \infty \right\}. \quad (4)$$

For any nonempty open set $\Omega \subset \mathbb{R}^n$ we have $n \leq b(\Omega) \leq \infty$.

If $\Omega$ is of finite measure, then $b(\Omega) = n$.

If $\Omega$ is unbounded and not quasi-bounded, then $b(\Omega) = \infty$. 
Examples

Let $\alpha > 0$, $\omega_\alpha, \Omega_\alpha \subset \mathbb{R}^2$ be as above. Then

\[
b(\omega_\alpha) = \begin{cases} 
\frac{1}{\alpha} + 1 & \text{if } 0 < \alpha < 1, \\
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There are quasi-bounded domains such that $b(\Omega) = \infty$. 
Quasi-bounded domains - compactness of embeddings

Theorem

(i) Let $b(\Omega) < \infty$. The embedding

$$
\bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega)
$$

is compact if

$$
\delta := s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} > \frac{b(\Omega)}{p^*} = b(\Omega)\left(\frac{1}{p_2} - \frac{1}{p_1}\right)_+.
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If the embedding (5) is compact and \( \frac{1}{p^*} = 0 \), then \( \delta > 0 \).
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If the embedding (5) is compact and \( \frac{1}{p^*} > 0 \), then \( \delta \geq \frac{b(\Omega)}{p^*} \).

(ii) If \( b(\Omega) = \infty \), then the embedding (5) is compact if, and only if, \( p_1 \leq p_2 \) and

\[
s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} > 0 . \tag{7}
\]
Theorem

Let $s_1 - s_2 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) > \frac{b(\Omega)}{p^*}$ and $b(\Omega) < \infty$. If

$$0 < \liminf_{j \to \infty} b_j(\Omega) 2^{-jb(\Omega)} \leq \limsup_{j \to \infty} b_j(\Omega) 2^{-jb(\Omega)} < \infty,$$

then

$$e_k\left(\bar{B}_{p_1,q_1}^{s_1}(\Omega) \hookrightarrow \bar{B}_{p_2,q_2}^{s_2}(\Omega)\right) \sim k^{-\gamma},$$

with

$$\gamma = \frac{s_1 - s_2}{b(\Omega)} + \frac{b(\Omega) - n}{b(\Omega)}\left(\frac{1}{p_1} - \frac{1}{p_2}\right).$$
Theorem

Let \( s_1 - s_2 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) > \frac{b(\Omega)}{p^*} \) and \( b(\Omega) < \infty \). If

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with \( \gamma = \frac{s_1 - s_2}{b(\Omega)} + \frac{b(\Omega) - n}{b(\Omega)}\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \).

Corollary

Let \( \Omega \) be of finite Lebesgue measure. If the embedding is compact, then

\[
\gamma = \frac{s_1 - s_2}{n}.
\]
What is the possible asymptotic behaviour of entropy numbers of the compact embedding of the function spaces defined on domains?
What is the possible asymptotic behaviour of entropy numbers of the compact embedding of the function spaces defined on domains?

**Theorem**

Let $s_1, s_2 \in \mathbb{R}$, $0 < p_1, p_2 \leq \infty$ and $0 < q_1, q_2 \leq \infty$. We assume that

$$\frac{s_1 - s_2}{n} > \left(\frac{1}{p_1} - \frac{1}{p_2}\right)^+.$$ 

For positive real $\gamma$, such that $\frac{s_1 - s_2}{n} \geq \gamma > \left(\frac{1}{p_1} - \frac{1}{p_2}\right)^+$, there exists a quasi-bounded domain $\Omega$ in $\mathbb{R}^n$ such that

$$e_k\left(\bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega)\right) \sim k^{-\gamma}, \quad k \in \mathbb{N}.$$  \hspace{1cm} (10)

If (10) holds for some quasi-bounded domain $\Omega$ in $\mathbb{R}^n$ and $b(\Omega) < \infty$, then

$$\frac{s_1 - s_2}{n} \geq \gamma > \left(\frac{1}{p_1} - \frac{1}{p_2}\right)^+.$$
Let \( A(x, D) = \sum_{|\alpha| \leq 2m} a_\alpha(x) \partial^\alpha \) be a formally self-adjoint, uniformly strongly elliptic differential operator of order \( 2m \), \( m \in \mathbb{N} \), with real valued coefficients \( a_\alpha \in C^\infty(\Omega) \) which are uniformly bounded and uniformly continuous for \( |\alpha| \leq 2m \). We assume that \( A \) is a positive self-adjoint operator in \( L_2(\Omega) \). Then \( A = A(x, D) \) is an operator with discrete spectrum \( \sigma(A) \) of eigenvalues having no finite accumulation point.
Let \( A(x, D) = \sum_{|\alpha| \leq 2m} a_\alpha(x) \partial^\alpha \) be a formally self-adjoint, uniformly strongly elliptic differential operator of order \( 2m \), \( m \in \mathbb{N} \), with real valued coefficients \( a_\alpha \in C^\infty(\Omega) \) which are uniformly bounded and uniformly continuous for \( |\alpha| \leq 2m \). We assume that \( A \) is a positive self-adjoint operator in \( L_2(\Omega) \). Then \( A = A(x, D) \) is an operator with discrete spectrum \( \sigma(A) \) of eigenvalues having no finite accumulation point.

**Theorem**

Let \( \Omega \) be quasi-bounded domain in \( \mathbb{R}^n \), such that \( b(\Omega) < \infty \) and (8) holds. Let \( \lambda_1, \lambda_2, \ldots \) be eigenvalues of \( A \) ordered by their magnitude and counted according to their multiplicities. Then

\[
\lambda_k \sim k^{\frac{2m}{b(\Omega)}}, \quad k \in \mathbb{N}.
\]
Examples

Let $\alpha > 0$ and $\alpha \neq 1$. For the open set $\Omega_\alpha \subset \mathbb{R}^2$ we have the following formula for the eigenvalues of the Dirichlet Laplacian

$$\lambda_k(-\Delta) \sim \begin{cases} k^{\frac{2\alpha}{1+\alpha}} & \text{if } 0 < \alpha < 1, \\ k^{\frac{2}{1+\alpha}} & \text{if } \alpha > 1. \end{cases} \quad (11)$$
Examples

Let $\alpha > 0$ and $\alpha \neq 1$. For the open set $\Omega_\alpha \subset \mathbb{R}^2$ we have the following formula for the eigenvalues of the Dirichlet Laplacian

$$
\lambda_k(-\Delta) \sim \begin{cases} 
k^{\frac{2\alpha}{1+\alpha}} & \text{if } 0 < \alpha < 1, \\
k^{\frac{2}{1+\alpha}} & \text{if } \alpha > 1.
\end{cases}
$$

The assumption (8) is sufficient but not necessary to get the estimates of corresponding entropy numbers. For the domain $\Omega_\alpha$ with $\alpha = 1$ one gets

$$
e_k\left(\bar{B}_{p_1,q_1}^{s_1}(\Omega_1) \hookrightarrow \bar{B}_{p_2,q_2}^{s_2}(\Omega_1)\right) \sim k^{-\frac{s_1-s_2}{2}} (\log k)^{\frac{s_1-s_2}{2} - \left(\frac{1}{p_1} - \frac{1}{p_2}\right)},$$

and

$$\lambda_k(-\Delta) \sim k \log k.$$
We regards the following weights

\[
 w_{(\alpha,\beta)}(x) = \begin{cases} 
 |x|^{\alpha_1}(1 - \log |x|)^{\alpha_2}, & \text{if } |x| \leq 1, \\
 |x|^{\beta_1}(1 + \log |x|)^{\beta_2}, & \text{if } |x| > 1,
\end{cases}
\] (12)

\[\alpha = (\alpha_1, \alpha_2), \quad \alpha_1 > -n, \quad \alpha_2 \in \mathbb{R}, \quad (13)\]

\[\beta = (\beta_1, \beta_2), \quad \beta_1 > -n, \quad \beta_2 \in \mathbb{R}.\] (13)
Muckenhaust weights and elliptic operators on $\mathbb{R}^n$

We regards the following weights

$$w_{(\alpha,\beta)}(x) = \begin{cases} |x|^\alpha (1 - \log |x|)^\alpha_2, & \text{if} \quad |x| \leq 1, \\ |x|^\beta_1 (1 + \log |x|)^\beta_2, & \text{if} \quad |x| > 1, \end{cases}$$ \hspace{1cm} (12)

$\alpha = (\alpha_1, \alpha_2)$, $\alpha_1 > -n$, $\alpha_2 \in \mathbb{R}$, $\beta = (\beta_1, \beta_2)$, $\beta_1 > -n$, $\beta_2 \in \mathbb{R}$. \hspace{1cm} (13)

This covers weights of purely polynomial growth both near 0 and $\infty$,

$$w_{\alpha,\beta}(x) \sim \begin{cases} |x|^\alpha & \text{if} \quad |x| \leq 1, \\ |x|^\beta & \text{if} \quad |x| > 1, \end{cases} \quad \text{with} \quad \alpha > -n, \quad \beta > -n.$$ \hspace{1cm} (14)
Degenerated elliptic operators on $\mathbb{R}^n$

Muckenhoupt weights and elliptic operators on $\mathbb{R}^n$

We regards the following weights

$$w_{(\alpha,\beta)}(x) = \begin{cases} |x|^{\alpha_1 (1 - \log |x|)^{\alpha_2}}, & \text{if } |x| \leq 1, \\ |x|^{\beta_1 (1 + \log |x|)^{\beta_2}}, & \text{if } |x| > 1, \end{cases} \quad (12)$$

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Problem: Criteria for compactness and entropy numbers of embeddings of type

$$\text{id} : B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n),$$

where $s_2 \leq s_1$, $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 \leq \infty$, 

L.Skrzypczak (UAM Poznań)
To estimate the negative spectrum \( \#\{\sigma_p(H_\gamma) \cap (-\infty, 0]\} \) of operators of type

\[ H_\gamma = A - \gamma V(x) \quad \text{as} \quad \gamma \to \infty , \]

where \( A \) is an elliptic pseudodifferential operator of order \( \kappa > 0 \), positive-definite and self-adjoint in \( L_2(\mathbb{R}^n) \), e.g. \( A = (\text{id} - \Delta)^{\kappa/2} \), and \( V(x) \) is a positive, real (singular) function.
Embeddings with Muckenhoupt weights - motivations

- To estimate the negative spectrum $\# \{ \sigma_p(H_\gamma) \cap (-\infty, 0] \}$ of operators of type
  
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- The Birman-Schwinger principle. Let $B = \sqrt{V} A^{-1} \sqrt{V}$ be compact.

  $$\sigma_e(H_\gamma) = \sigma_e(A) \quad \text{and} \quad \# \{ \sigma(H_\gamma) \cap (-\infty, 0] \} = \# \{ \sigma_p(H_\gamma) \cap (-\infty, 0] \} = \# \{ k \in \mathbb{N} : \mu_k(B) \geq 1 \} < \infty.$$
Embeddings with Muckenhoupt weights - motivations

- To estimate the negative spectrum $\#\{\sigma_p(H_\gamma) \cap (-\infty, 0]\}$ of operators of type
  \[ H_\gamma = A - \gamma V(x) \quad \text{as} \quad \gamma \to \infty, \]
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  \[ = \#\{k \in \mathbb{N} : \mu_k(B) \geq 1\} < \infty. \]

- By factorization we can reduce the estimates for $B$ to the estimates for the Sobolev embeddings of weighted spaces.
Degenerated elliptic operators on $\mathbb{R}^n$

Negative spectrum - purely polynomial estimates for $V$.

Theorem

Let $\kappa > 0$, $w_{\alpha, \beta}V \in L_{\infty}(\mathbb{R}^n)$, and

$$-n < \alpha < n, \quad 0 < \beta < n, \quad \kappa > \alpha_+, \quad \kappa \neq \beta.$$  \hspace{1cm} (15)

Then

$$\#\{\sigma_p(H_{\gamma}) \cap (-\infty, 0]\} \leq c \left( \gamma \| w_{\alpha, \beta}V \|_{L_{\infty}}^2 \right)^{\frac{n}{\min(\kappa, \beta)}}, \quad \gamma \to \infty.$$  \hspace{1cm} (16)
Degenerated elliptic operators on $\mathbb{R}^n$

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Then

$$\#\{\sigma_p(H_\gamma) \cap (-\infty, 0]\} \leq c (\gamma \|w_{\alpha,\beta} V\|_{L_\infty}^2)^{\frac{n}{\min(\kappa,\beta)}}, \quad \gamma \to \infty. \quad (16)$$

**Examples**

Let $H_\gamma = A - \gamma |x|^{-\mu}$, $0 < \mu < n$, $\kappa > \mu$, i.e. $V(x) = |x|^{-\mu}$.

Then

$$\#\{\sigma_p(H_\gamma) \cap (-\infty, 0]\} \leq C \gamma^{\frac{n}{\mu}}, \quad \gamma \to \infty.$$
Degenerated elliptic operators on $\mathbb{R}^n$  

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**Examples**

Let $H_{\gamma} = A - \gamma |x|^{-\mu}$, $0 < \mu < n$, $\kappa > \mu$, i.e. $V(x) = |x|^{-\mu}$.

Then

$$\# \{ \sigma_p(H_{\gamma}) \cap (-\infty, 0] \} \leq C \gamma^{\frac{n}{\mu}}, \quad \gamma \to \infty.$$  

Limiting cases: $\kappa = \beta > 0$ or $\kappa = \alpha > 0$?
Negative spectrum - limiting cases

Theorem

Let $0 < \kappa < n$ and $w_{(\alpha,\beta)}V \in L_\infty(\mathbb{R}^n)$. 
Degenerated elliptic operators on $\mathbb{R}^n$

Negative spectrum - limiting cases

Theorem

Let $0 < \kappa < n$ and $w_{(\alpha, \beta)} V \in L_\infty(\mathbb{R}^n)$.

(i) Let $\alpha_1 = \kappa$, $\alpha_2 > 2 \frac{\kappa}{n}$, $0 < \beta_1 = \beta < n$ and $\beta_2 = 0$. Then there exists a positive constant $C > 0$ independent of $\gamma$, $\beta$ and $V$ such that

$$\# \{ \sigma_p(H_\gamma) \cap (-\infty, 0] \} \leq C \begin{cases} \gamma^{\frac{n}{\kappa}} & \text{if } \kappa < \beta, \\ \gamma^{\frac{n}{\beta}} & \text{if } \kappa > \beta, \\ \gamma^{\frac{n}{\kappa}} \log \gamma & \text{if } \kappa = \beta, \end{cases} \gamma \to \infty.$$
Degenerated elliptic operators on $\mathbb{R}^n$

Negative spectrum - limiting cases

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Let $0 < \kappa < n$ and $w_{(\alpha, \beta)} V \in L_\infty(\mathbb{R}^n)$.

(i) Let $\alpha_1 = \kappa$, $\alpha_2 > 2\frac{\kappa}{n}$, $0 < \beta_1 = \beta < n$ and $\beta_2 = 0$. Then there exists a positive constant $C > 0$ independent of $\gamma$, $\beta$ and $V$ such that

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(ii) Assume $\max(0, \alpha_1) < \kappa$ and $\beta_1 = \kappa$. Then there exists a positive constant $C > 0$ independent of $\gamma$, $\beta$ and $V$ such that

$$\# \{ \sigma_p(H_{\gamma}) \cap (-\infty, 0] \} \leq C \begin{cases} \gamma^{\frac{n}{\kappa}} & \text{if } \frac{\kappa}{n} < \beta_2, \\ \gamma^{\frac{n}{\kappa}} (\log \gamma)^{1-\beta_2 \frac{n}{\kappa}} & \text{if } 0 \geq \beta_2, \end{cases} \gamma \to \infty.$$
Examples

(i) Let $0 < \kappa < n$, $\varepsilon > 0$, and

$$V(x) = \begin{cases} |x|^{-\kappa}(1 - \log |x|)^{-\frac{2\kappa}{n} - \varepsilon} & \text{if } |x| < 1, \\ |x|^{-\kappa}(1 + \log |x|)^{-\beta} & \text{if } |x| \geq 1. \end{cases}$$
Examples

(i) Let $0 < \kappa < n$, $\varepsilon > 0$, and

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\end{cases}
\]

\[\#\{\sigma_p(H_\gamma) \cap (-\infty, 0]\} \leq C \begin{cases} 
\gamma^\frac{n}{\kappa}, & \text{if } \beta > \frac{\kappa}{n}, \\
\gamma^\frac{n}{\kappa}(\log \gamma)^{\frac{n}{\kappa}(-\beta)} & \text{if } \beta < \frac{\kappa}{n}.
\end{cases}\]
Examples

(i) Let $0 < \kappa < n$, $\epsilon > 0$, and

$$V(x) = \begin{cases} |x|^{-\kappa}(1 - \log |x|)^{-2\frac{\kappa}{n} - \epsilon} & \text{if } |x| < 1, \\ |x|^{-\kappa}(1 + \log |x|)^{-\beta} & \text{if } |x| \geq 1. \end{cases}$$

Then

$$\#\{\sigma_p(H_\gamma) \cap (-\infty, 0]\} \leq C \begin{cases} \gamma \frac{n}{\kappa}, & \text{if } \beta > \frac{\kappa}{n}, \\ \gamma \frac{n}{\kappa} (\log \gamma)^{1 + \frac{n}{\kappa} (-\beta)}_+ & \text{if } \beta < \frac{\kappa}{n}. \end{cases}$$

(ii) Let

$$V(x) = \begin{cases} (1 - \log |x|)^{\alpha} & \text{if } |x| < 1, \\ (1 + \log |x|)^{-\alpha} & \text{if } |x| \geq 1, \end{cases}, \quad \alpha > 0, \quad \kappa > 0.$$  

Then

$$\#\{\sigma_p(H_\gamma) \cap (-\infty, 0]\} \leq C \exp \left(\gamma \frac{1}{\alpha}\right), \quad \gamma \to \infty.$$
Let us regard the operator $B = b_2 A^{-1} b_1$ with

$$b_1 w_{(0,\beta)} \in L_{r_1}(\mathbb{R}^n), \quad b_2 w_{(0,\eta)} \in L_{r_2}(\mathbb{R}^n) \quad \text{where} \quad \beta_1 = \eta_1 = 0,$$

$$1 \leq r'_1 < p < r_2 \leq \infty, \quad 1 < p < \infty, \quad \text{and} \quad \kappa > n \left( \frac{1}{r_1} + \frac{1}{r_2} \right).$$

If $\beta_2 + \eta_2 > 0$ then the operator $B$ is compact in $L_p(\mathbb{R}^n)$ and

$$|\mu_k(B)| \leq c \|b_1 w_{(0,\beta)}\|_{L_{r_1}(\mathbb{R}^n)} \|b_2 w_{(0,\eta)}\|_{L_{r_2}(\mathbb{R}^n)} \times$$

$$\begin{cases} k^{-\beta_2-\eta_2} & \text{if} \quad \beta_2 + \eta_2 \leq \frac{1}{r_1} + \frac{1}{r_2} \\
 k^{-\frac{1}{r_1}-\frac{1}{r_2}} (1 + \log k)^{-\beta_2-\eta_2 + \frac{1}{r_1} + \frac{1}{r_2}} & \text{if} \quad \beta_2 + \eta_2 > \frac{1}{r_1} + \frac{1}{r_2}. \end{cases}$$

(17)
The last statement can be reduced by the Hölder inequality and the Carl inequality to the estimates of entropy numbers of Sobolev embeddings

\[ \text{id} : B^{s_1}_{p_1, q_1}(\mathbb{R}^n, w_{(0,\beta)}) \hookrightarrow B^{s_2}_{p_2, q_2}(\mathbb{R}^n), \quad \text{with} \quad \beta_1 = 0 \quad \text{and} \]

\[
\frac{1}{p_1} = \frac{1}{p} + \frac{1}{r_1}, \quad \frac{1}{p} = \frac{1}{p_2} + \frac{1}{r_2}.
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In case of \( \delta = s_1 - \frac{1}{p_1} - s_2 + \frac{1}{p_2} > 0, \beta_2 > 0, \) then for all \( k \in \mathbb{N}, \)

\[ e_k(\text{id}) \sim \begin{cases} 
  k^{-\frac{\beta_2}{p_1}}, & \text{if } \frac{\beta_2}{p_1} \leq \frac{1}{p_1} - \frac{1}{p_2}, \\
  k^{-\frac{1}{p_1} + \frac{1}{p_2}} (1 + \log k)^{-\frac{\beta_2}{p_1} + \frac{1}{p_1} - \frac{1}{p_2}}, & \text{if } \frac{\beta_2}{p_1} > \frac{1}{p_1} - \frac{1}{p_2}.
\end{cases} \]
**Logarithmic type weights - entropy numbers**

The last statement can be reduced by the Hölder inequality and the Carl inequality to the estimates of entropy numbers of Sobolev embeddings

\[ \text{id} : B_{p_1,q_1}^{s_1}(\mathbb{R}^n, w(0,\beta)) \hookrightarrow B_{p_2,q_2}^{s_2}(\mathbb{R}^n), \quad \text{with} \quad \beta_1 = 0 \quad \text{and} \]

\[
\frac{1}{p_1} = \frac{1}{p} + \frac{1}{r_1}, \quad \frac{1}{p} = \frac{1}{p_2} + \frac{1}{r_2}.
\]

In case of \( \delta = s_1 - \frac{1}{p_1} - s_2 + \frac{1}{p_2} > 0, \beta_2 > 0, \) then for all \( k \in \mathbb{N}, \)

\[ e_k(\text{id}) \sim \begin{cases} 
  k^{-\frac{\beta_2}{p_1}}, & \text{if} \quad \frac{\beta_2}{p_1} \leq \frac{1}{p_1} - \frac{1}{p_2}, \\
  k^{-\frac{1}{p_1} + \frac{1}{p_2}} (1 + \log k)^{-\frac{\beta_2}{p_1} + \frac{1}{p_1} - \frac{1}{p_2}}, & \text{if} \quad \frac{\beta_2}{p_1} > \frac{1}{p_1} - \frac{1}{p_2}.
\end{cases} \]

Alternative strategy: \( x_k(\text{id}) \sim ? \) plus the Pietsch inequality.
Theorem

Let $0 < p_1 \leq p_2 \leq \infty$, $1 < q_1, q_2 \leq \infty$ and $s_1 - s_2 - d\left(\frac{1}{p_1} - \frac{1}{p_2}\right) > 0$. Then

$$x_k(\text{id}) \sim (1 + \log k)^{-\frac{\beta_2}{p_1}} \begin{cases} 
  k^{-\left(\frac{1}{p_1} - \frac{1}{p_2}\right)} & \text{if } 1 \leq p_1 \leq p_2 \leq 2, \\
  k^{-\left(\frac{1}{p_1} - \frac{1}{2}\right)} & \text{if } 1 \leq p_1 < 2 < p_2 \leq \infty, \\
  1 & \text{if } 2 \leq p_1 \leq p_2 \leq \infty.
\end{cases}$$
Logarithmic type weights - Weyl numbers

Theorem

Let \(0 < p_1 \leq p_2 \leq \infty\), \(1 < q_1, q_2 \leq \infty\) and \(s_1 - s_2 - d(\frac{1}{p_1} - \frac{1}{p_2}) > 0\). Then

\[
x_k(\text{id}) \sim (1 + \log k)^{-\frac{\beta_2}{p_1}} \begin{cases} 
  k^{-\left(\frac{1}{p_1} - \frac{1}{p_2}\right)} & \text{if } 1 \leq p_1 \leq p_2 \leq 2, \\
  k^{-\left(\frac{1}{p_1} - \frac{1}{2}\right)} & \text{if } 1 \leq p_1 < 2 < p_2 \leq \infty, \\
  1 & \text{if } 2 \leq p_1 \leq p_2 \leq \infty.
\end{cases}
\]

Remark

One can show also that

\[
a_k(\text{id}) \sim c_k(\text{id}) \sim d_k(\text{id}) \sim (1 + \log k)^{-\frac{\beta_2}{p_1}}.
\]
Let us regard once more the operator $B = b_2 A^{-1} b_1$ with

$$b_1 w_{(0,\beta)} \in L_{r_1}(\mathbb{R}^n), \quad b_2 w_{(0,\eta)} \in L_{r_2}(\mathbb{R}^n) \quad \text{where} \quad \beta_1 = \eta_1 = 0,$$

$$1 \leq r'_1 < p < r_2 \leq \infty, \quad 1 < p < \infty, \quad \text{and} \quad \kappa > n \left( \frac{1}{r_1} + \frac{1}{r_2} \right).$$

If $\beta_2 + \eta_2 > 0$ then the operator $B$ is compact in $L_p(\mathbb{R}^n)$ and

$$|\mu_k(B)| \leq c \left\| b_1 w_{(0,\beta)} |L_{r_1}(\mathbb{R}^n)| \right\| \left\| b_2 w_{(0,\eta)} |L_{r_2}(\mathbb{R}^n)| \right\| \times \quad (18)$$

$$\begin{cases} (1 + \log k)^{-\beta_2 - \eta_2} k^{-\frac{1}{r_1} - \frac{1}{r_2}} & \text{if} \quad \frac{1}{2} + \frac{1}{r_2} \leq \frac{1}{p}, \\
(1 + \log k)^{-\beta_2 - \eta_2} k^{-\frac{1}{p} - \frac{1}{r_1} + \frac{1}{2}} & \text{if} \quad \frac{1}{2} - \frac{1}{r_1} < \frac{1}{p} < \frac{1}{2} + \frac{1}{r_2} \quad \text{and} \quad \beta_2 + \eta_2 \leq \frac{1}{p} + \frac{1}{r_1} - \frac{1}{2}, \end{cases}$$