Growth envelopes in Muckenhoupt weighted function spaces

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Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$. The **Hardy-Littlewood maximal operator** M is given by

$$Mf(x) := \sup_{B \in \mathcal{B}, x \in B} \frac{1}{|B|} \int_B |f(y)| dy$$

where $\mathcal{B}$ is the collection of all open balls.

Definition

Let $w \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ positive a.e.
We say w belongs to the **Muckenhoupt class** $\mathcal{A}_1$, if there exists an $0 < A < \infty$ such that for almost all $x \in \mathbb{R}^n$:

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- Muckenhoupt 1972, 1973
- $w(\Omega) = \int\limits_{\Omega} w(x)dx, \quad \Omega \subset \mathbb{R}^n$ bounded, measurable set.

Example

$$w_{\alpha,\beta}(x) = \begin{cases} |x|^{\alpha}, & \text{if} \quad |x| \leq 1, \\ |x|^{\beta}, & \text{if} \quad |x| > 1, \end{cases} \qquad \alpha,\ \beta > -n.$$

$$w_{\alpha,\beta} \in \mathcal{A}_1 \quad \Leftrightarrow \quad -n < \alpha, \beta \leq 0.$$

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1.2 Weighted Besov- and Triebel-Lizorkin spaces

Let $w \in \mathcal{A}_1$ a Muckenhoupt weight and $0 < p < \infty$. The weigthed Lebesgue space $L_p(w) = L_p(\mathbb{R}^n, w)$ contains all measurable functions such that

$$\|f|L_p(w)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)dx\right)^{1/p}$$

is finite.

Definition (Bui '82)

Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and $\{\varphi_j\}_j$ a smooth dyadic resolution of unity. Assume $w \in \mathcal{A}_1$.

(i) The weighted Besov space $B^s_{p,q}(w)$ is the set of all distributions $f \in \mathcal{S}'$ such that

$$\left\| f | B^s_{p,q}(w) \right\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) | L_p(w) \right\|^q \right)^{1/q}$$

is finite, (with the usual modification in the limiting case $q = \infty$).

(ii) The weighted Triebel-Lizorkin space $F^s_{p,q}(w)$ is the set of all distributions $f \in \mathcal{S}'$ such that

$$\left\| f | F^s_{p,q}(w) \right\| = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} \left| \mathcal{F}^{-1}(\varphi_j \mathcal{F} f)(\cdot) \right|^q \right)^{1/q} | L_p(w) \right\|$$

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1.3 Growth envelopes

Let for some measurable function $f : \mathbb{R}^n \to \mathbb{C}$, finite a.e., its **decreasing rearrangement** f^* be defined as usual,

$$f^*(t) := \inf\{s > 0 : |\{x \in \mathbb{R}^n : |f(x)| > t\}| \le t\}, \quad t \ge 0.$$

Definition

Let X be some quasi-normed function space on $\mathbb{R}^n$. The **growth envelope function** $\mathcal{E}_G^X : (0, \infty) \to [0, \infty]$ of X is defined by

$$\mathcal{E}_G^X(t) := \sup_{f \in X, ||f|X|| \le 1} f^*(t), \qquad t > 0.$$

Proposition

Let X, X_1, X_2 be (quasi-)normed function spaces on $\mathbb{R}^n$. Then

(i) $\mathcal{E}_{\mathsf{G}}^{X}$ *is monotonically decreasing and right-continuous,* $(\mathcal{E}_{\mathsf{G}}^{X})^* = \mathcal{E}_{\mathsf{G}}^{X}$.

(ii)

$$X \hookrightarrow L_\infty \quad \Longleftrightarrow \quad \mathcal{E}_{\mathsf{G}}^{X}(\cdot) \quad \text{is bounded.}$$

(iii)

$$X_1 \hookrightarrow X_2 \quad \Longrightarrow \quad \exists\, c > 0 \quad \forall\, t > 0 : \quad \mathcal{E}_{\mathsf{G}}^{X_1}(t) \le c\, \mathcal{E}_{\mathsf{G}}^{X_2}(t).$$

- „fine index“ u_{G}^{X} $\rightsquigarrow$ $\mathfrak{E}_{\mathsf{G}}(X) = \left(\mathcal{E}_{\mathsf{G}}^{X}(\cdot), u_{\mathsf{G}}^{X}\right)$ **growth envelope**
- $\mathfrak{E}_{\mathsf{G}}(L_{p,q}) = \left(t^{-\frac{1}{p}}, q\right)$, $0 < p < \infty$, $0 < q \le \infty$, $\mathfrak{E}_{\mathsf{G}}(L_p) = \left(t^{-\frac{1}{p}}, p\right)$.

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2.1 Growth envelope for $B^s_{p,q}(w)$, $F^s_{p,q}(w)$, $w \in \mathcal{A}_1$

Main Theorem

Let $0 < p < \infty$, $0 < q \leq \infty$, $n \max(\frac{1}{p} - 1, 0) < s < \frac{n}{p}$ *and* $w \in \mathcal{A}_1$ *with*

$$\inf_{m \in \mathbb{Z}^n} w(Q_{0,m}) \geq c > 0.$$

Then

(i)

$$\mathfrak{E}_{\mathsf{G}}(B^s_{p,q}(w)) = \mathfrak{E}_{\mathsf{G}}(B^s_{p,q}) = \left(t^{-\frac{1}{p}+\frac{s}{n}}, q\right).$$

(ii)

$$\mathfrak{E}_{\mathsf{G}}(F^s_{p,q}(w)) = \mathfrak{E}_{\mathsf{G}}(F^s_{p,q}) = \left(t^{-\frac{1}{p}+\frac{s}{n}}, p\right).$$

(iii)

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w)}(t) \sim \mathcal{E}_{\mathsf{G}}^{F^s_{p,q}(w)}(t) \sim \mathcal{E}_{\mathsf{G}}^{B^s_{p,q}}(t) \sim \mathcal{E}_{\mathsf{G}}^{F^s_{p,q}}(t) \sim t^{-\frac{1}{p}}, \qquad t \to \infty.$$

2.2.1 Atoms

Let for $m \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$, $Q_{\nu,m}$ denote the n-dimensional cube with sides parallel to the axes of coordinates, centered at $2^{-\nu}m$ and with side length $2^{-\nu}$.

Definition

Let $0 < p < \infty$, $0 < q \leq \infty$ and $w \in \mathcal{A}_1$.

$$b_{p,q}(w) = \Bigg\{ \lambda = \{\lambda_{\nu,m}\}_{\nu,m} : \lambda_{\nu,m} \in \mathbb{C},$$

$$||\lambda|b_{p,q}(w)|| = \left(\sum_{\nu=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu,m}|^p \, 2^{\nu n} \, w(Q_{\nu,m}) \right)^{q/p} \right)^{1/q} < \infty \Bigg\}$$

(with the usual modification in the limiting case $q = \infty$)

Definition

Let $s \in \mathbb{R}$, $0 < p \leq \infty$, $K \in \mathbb{N}_0$ and $d > 1$.

The complex-valued function $a \in C^K(\mathbb{R}^n)$ is said to be an $(s,p)_K$-atom, if for some $\nu \in \mathbb{N}_0$ and some $m \in \mathbb{Z}^n$ holds

$$\operatorname{supp}\, a \subset d\, Q_{\nu,m} \quad \text{and}$$
$$|D^\alpha a(x)| \leq 2^{-\nu(s-\frac{n}{p})+|\alpha|\nu} \quad \text{for} \quad |\alpha| \leq K, \quad x \in \mathbb{R}^n.$$

Let $\alpha_+ := \max(\alpha, 0)$ and $\lfloor \alpha \rfloor := \max\{k \in \mathbb{Z} : k \le \alpha\}$

Proposition (atomic decomposition)

Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, $K \in \mathbb{N}_0$ and $w \in \mathcal{A}_1$ with

$$K \ge (1 + \lfloor s \rfloor)_+ \quad \text{and} \quad n\left(\frac{1}{p} - 1\right)_+ < s;$$

then $f \in \mathcal{S}'$ belongs to $B^s_{p,q}(w)$ if, and only if, it can be written as

$$f(x) = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m}(x), \quad \text{converging in } \mathcal{S}',$$

where $a_{\nu,m}$ are $(s,p)_K$-atoms and $\lambda = \{\lambda_{\nu,m}\}_{\nu,m} \in b_{p,q}(w)$; furthermore

$$||f|B^s_{p,q}(w)|| \sim \inf ||\lambda|b_{p,q}(w)||$$

where the infimum is taken over all admissible representations.

Estimate from below: atomic decomposition

$$f_\lambda(x) = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m}(x),$$

with special $\lambda = \{\lambda_{\nu,m}\}_{\nu,m} \in \mathbb{C}$ and $a_{\nu,m}$ are atoms. Then:

$$||f_\lambda|B^s_{p,q}(w)|| \leq ||\lambda|b_{p,q}(w)||$$

Hence

$$\begin{aligned}\mathcal{E}_G^{B^s_{p,q}(w)}(t) &= \sup_{f \in B^s_{p,q}(w), ||f|B^s_{p,q}(w)|| \leq 1} f^*(t) \\ &\geq c \sup_{f_\lambda \in B^s_{p,q}(w), ||f_\lambda|B^s_{p,q}(w)|| \leq 1} f_\lambda^*(t) \geq f^*_{\lambda_0}(t) \sim t^{-\frac{1}{p}+\frac{s}{n}},\end{aligned}$$

pointwise for a fixed $t \in (0,1)$, $\lambda_0 = \lambda(t)$.

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2.2.2 Embeddings

For $0 < p_1, p_2 \leq \infty$ and $s_1, s_2 \in \mathbb{R}$ we adopt the notation $\delta := s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2}$.

Corollary

Let $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 \leq \infty$, $-\infty < s_2 \leq s_1 < \infty$ *and* $w \in \mathcal{A}_1$.

Then the embedding $B^{s_1}_{p_1,q_1}(w) \hookrightarrow B^{s_2}_{p_2,q_2}$ *is continuous if, and only if,*

(a) $$\inf_l w(Q_{0,l}) \geq c > 0,$$

(b) $$\begin{cases} \delta > 0, & \text{if} \quad q_1 > q_2 \\ \delta \geq 0, & \text{if} \quad q_1 \leq q_2 \end{cases},$$

(c) $$p_1 \leq p_2.$$

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Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ *and* $w \in \mathcal{A}_1$. *Then*

(i)

$$B^s_{p,\min(p,q)}(w) \hookrightarrow F^s_{p,q}(w) \hookrightarrow B^s_{p,\max(p,q)}(w).$$

(ii) *Assume*

$$\inf_{m \in \mathbb{Z}^n} w(Q_{0,m}) \geq c > 0.$$

Let $0 < p_0 < p < p_1 < \infty$, $-\infty < s_1 < s < s_0 < \infty$ *satisfy*

$$s_0 - \frac{n}{p_0} = s - \frac{n}{p} = s_1 - \frac{n}{p_1}.$$

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Applications

We have

$$\mathfrak{E}_{\mathsf{G}}(B^s_{p,q}(w)) = \left(t^{-\frac{1}{p}+\frac{s}{n}}, q\right)$$

this means

$$\left(\int_0^{\epsilon} \left[t^{\frac{1}{p}-\frac{s}{n}} f^*(t)\right]^q \frac{dt}{t}\right)^{1/q} \leq c \,\|f|B^s_{p,q}(w)\|$$

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Then for $0 < u, v \leq \infty$,

$$B^{s_0}_{p_0,u}(w) \hookrightarrow F^{s}_{p,q}(w) \hookrightarrow B^{s_1}_{p_1,v}(w) \quad \textit{if and only if} \quad u \leq p \leq v.$$

Corollary (continued)

(ii) *Let* $1 < r < \infty$, $0 < u \leq \infty$, $n\left(\frac{1}{p} - 1\right)_+ < s < \frac{n}{p}$, *with*

$$s - \frac{n}{p} = -\frac{n}{r}.$$

Then

$$B^s_{p,q}(w) \hookrightarrow L_{r,u} \quad \text{if and only if} \quad q \leq u \leq \infty$$

$$F^s_{p,q}(w) \hookrightarrow L_{r,u} \quad \text{if and only if} \quad p \leq u \leq \infty$$

Outlook

- more results for special weights, e.g.: $w_{\alpha,\beta}$

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Thank you for your attention!