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Approximating of linear combination of different shifts by argument functions defined on real axis by means of Valle Poussin's operations

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1 The article is devoted to research of problems concerning approximation of sums of $\sum_{i=1}^{m} \alpha_i(f(x+\delta_i)$ *kind of the entire functions exponential type. Are received asimptotic laws of behaviour of the upper bounded of deviations of indicated sums in the uniform metrics.*

As a rule trigonometric polynominals of given degree n are used as approximating aggregates for periodic functions, and the partial sums of decomposing the function into Fourier series serves as a simplest example of such aggregates. Though it is well-known that there exist the continuous functions, whose Fourier series are diverge in separate points. Therefore linear processes of summation of Fourier series have been developed thanks to which, starting from decomposition of functions into Fourier series, we can obtain sequences of operators which are uniformly convergent on the whole $C_{2\pi}$ continuous 2π-periodic functions.

For the approximation of functions defined on the real axis and not being necessarily periodic entire functions of exponential type not greater than σ are the natural device.

The beginning of the modern theory of approximation by entire functions was initiate in the works of Bernshteins at the beginning of the last century. Bernshtein was the proposer of the idea creating a theory of approximation of functions defined on the real axis which would include the approximation theory of periodic functions. This idea became very important for both theories. They have developed in parallel over the past decades mutually enriching and complementing each other. This work is in the channel of this concept.

In 1983 O.I. Stepanets [1]offered new classification of periodic functions with the help of multipliers and shifting arguments. This classification allowed considering all spectrum of summable functions and more subtle properties of separate function.

In 1996 appeared a paper O.I. Stepanets, Wang Kunyang and Zhang Xirong [1] which introduced the concept of generalised $\overline{\psi}$-derivatives of functions which are locally integrable on the real axis (and not necessarily periodic) and defined classes $\widehat{C}^{\overline{\psi}}_{\infty}$, $\widehat{C}^{\overline{\psi}}H_{\omega}$. These classes generalized classes introduced earlier and contain, as a special case, classes of continuous periodic functions. Here we give their definition.

Let [2] $\overline{\psi} = (\psi_1, \psi_2)$ be pair of functions $\psi_1(v)$, $\psi_2(v)$ such that $\psi_1 \in \mathfrak{A}$, $\psi_2 \in \mathfrak{A}'$, where $\mathfrak{A}$ is a set of the continuous at $v \geq 0$ functions $\psi(v)$, which obey the following:

1) $\psi(v) \geq 0$, $\psi(0) = 0$, $\psi(v)$ increases on $[0, 1)$;

2) $\psi(v)$ is a convex downward function on $[1, \infty)$ and such that $\lim\limits_{v\to\infty} \psi(v) = 0$;

3) the derivative $\psi'(v) = \psi'(v + 0)$ be a function of bounded variation in the interval $[0, \infty)$.

$\mathfrak{A}'$ is a subset of functions $\psi \in \mathfrak{A}$, such that

$$\int\limits_1^{\infty} \frac{\psi(v)}{v}\, dv < \infty.$$

Let, further, C be a set of continuous bounded on the real axis functions, $\mathfrak{N}$ is a unit ball S_∞ in the space M :

$$S_\infty = \{\varphi : \operatorname{ess\,sup} |\varphi(t)| \leq 1\},$$

or the class

$$H_\omega = \{\varphi \in C : |\varphi(t) - \varphi(t')| \le \omega(|t - t'|) \ \forall t, t' \in \mathbb{R}\},$$

where $\omega(t)$ is a fixed modulus of continuity. Then, according to Stepanets [2,3], through $\widehat{C}^{\overline{\psi}}\mathfrak{N}$ we denote the set of continuous functions f admitting for all x the representations

$$f(x) = A_0 + \int\limits_{\mathbb{R}} \varphi(x+t)\widehat{\overline{\psi}}(t)\, dt \stackrel{\mathrm{df}}{=} A_0 + \varphi * \widehat{\overline{\psi}}\ (x), \tag{1}$$

where A_0 is a some constant and the integral is understood as the limit of integrals over symmetric expanding segments, function $\varphi \in \mathfrak{N}$,

$$\widehat{\overline{\psi}}(t) = \frac{1}{2\pi} \int\limits_{\mathbb{R}} (\psi_{1+}(x) + i\psi_{2-}(x))e^{-ixt}\, dx, \tag{2}$$

and ψ_{1+}, ψ_{2-} even and odd extanding of functions ψ_1, ψ_2 accordingly. If $\psi_1 \in \mathfrak{A}$, $\psi_2 \in \mathfrak{A}'$, the transformation $\widehat{\overline{\psi}}(t)$ is summable on the entire axis (see, for example, [3], the Proposition 9.5.1). Function $\varphi(\cdot)$ in the representations (1) is named $\overline{\psi}$-derivative of function $f(\cdot)$ and denoted $f^{\overline{\psi}}(\cdot)$.

Every functions $f \in \widehat{C}^{\overline{\psi}}\mathfrak{N}$ at all real $\sigma > h \ge 1$ is associating with the operator $V_{\sigma,h}(f;x)$, having put

$$V_{\sigma,h}(f;x) = A_0 + f^{\overline{\psi}} * \widehat{\lambda_{\sigma,h}\overline{\psi}}\ (x), \tag{3}$$

where $\widehat{\lambda_{\sigma,h}\overline{\psi}}$ is a transformation kind (2) of the functions $\lambda_{\sigma,h}(t)\overline{\psi}(t)$, in which

$$\lambda_{\sigma,h}(t) = \begin{cases} 1, & 0 \le |t| \le \sigma - h, \\ \frac{\sigma - |t|}{h}, & \sigma - h \le |t| \le \sigma, \\ 0, & \sigma \le |t|. \end{cases} \tag{4}$$

Such operators were considered Stepanets in the works [2, 3], where it is shown (see for example [3], the Proposition 9.3.4) that when satisfying conditions imposed above on function ψ_1, ψ_2 and $f \in \widehat{C}^{\overline{\psi}}\mathfrak{N}$ operators $V_{\sigma,h}(f;x)$ belong to set ε_σ the set of all entire functions exponential type not greater than a given number σ. In the periodic case if σ and h is a natural numbers $V_{\sigma,h}(f;x)$ coincide with the well-known Valle Poussin's sums

$$V_{n,p}(f;x) = \frac{1}{p}\sum_{k=n-p}^{n-1} S_k(f;x),$$

where $S_k(f;x),\ k = 0, 1, \ldots,$ are partial Fourier sums of the function f of order k. Therefore $V_{\sigma,h}(f;x)$ are named as Valle Poussin's operators.

The subject of the researches are the sums

$$\Sigma_{\sigma,h,m}^{\alpha,\delta}(f;x) = \sum_{i=1}^{m} \alpha_i(f(x+\delta_i) - V_{\sigma,h}(f;x+\delta_i)), \qquad (5)$$

where $\alpha_i = \alpha_i(\sigma)$, $\delta_i = \delta_i(\sigma)$ are quantities uniformly bounded in σ. We find asymptotic formulas as $\sigma \to \infty$ for quantities

$$\Sigma_{\sigma,h,m}^{\alpha,\delta}(\widehat{C}^{\overline{\psi}}\mathfrak{N}) = \sup_{f\in\widehat{C}^{\overline{\psi}}\mathfrak{N}} ||\Sigma_{\sigma,h,m}^{\alpha,\delta}(f;x)||_C. \qquad (6)$$

The given problem has rich history. It originates in Bernstein's works (see [4], pp. 446 - 467) which studying properties the entire functions of finite degree has proved that linear combinations $f(x+x_0) \pm f(x-x_0)$ can be bounded on the real axes for some unlimited functions $f(x)$:

$$\frac{1}{2}\left|f(x-\frac{\pi}{2\sigma}) + f(x+\frac{\pi}{2\sigma})\right| \le \frac{4}{\pi}\sup_k\left|f\left(\frac{k\pi}{\sigma}\right)\right|, \qquad k \in \mathbb{Z}. \qquad (7)$$

14This phenomenon was named by him as interference.

Further these questions studied Akhiezer, Timan [5], Boas [6], Macintyre, Drozd [7] and others. In particular, Timan obtained the inequality

$$\frac{1}{2}\left|f(x-\frac{m\pi}{2\sigma})+f(x+\frac{m\pi}{2\sigma})\right|\leq\frac{4M}{\pi m}+\frac{8mM}{\pi}\sum_{i=1}^{\frac{m-1}{2}}\frac{1}{m^2-4i^2}, \qquad (8)$$

where m any odd number, $M=\sup\limits_{k}\left|f\left(\frac{k\pi}{\sigma}\right)\right|$, $k\in\mathbb{Z}$.

The set $\mathfrak{A}$ is very inhomogeneous in the rate of convergence of its elements to zero as $t\to\infty$. Therefore, these arises the necessity of decomposing the set $\mathfrak{A}$ into subset of function $\psi\in\mathfrak{A}$ with the same, in a certain sense, character of convergence to zero.

For this purpose each function $\psi\in\mathfrak{A}$ $\forall t\geq 1$ we will connecting with the pair of functions $\eta(t)=\psi^{-1}(\psi(t)/2))$ and $\mu(t)=t/(\eta(t)-t)$. We select the following subsets of the set $\mathfrak{A}$:

$$\mathfrak{A}_C=\{\psi\in\mathfrak{A}:0<K_1\leq\mu(t)\leq K_2<\infty\},$$

$$\mathfrak{A}_\infty^+=\{\psi\in\mathfrak{A}:\mu(t)\uparrow\infty\},$$

$$\overline{F}=\{\psi\in\mathfrak{A}:\eta'(t)\leq K_3\},$$

where K_j, $j=\overline{1,3}$ are the some constants, which, probably, depend of the function $\psi(t)$. The set $\overline{F}$ contains the functions that cannot decrease to zero more slowly some negative degree t. Note that

$$(\mathfrak{A}_C\cup\mathfrak{A}_\infty^+)\subset\overline{F}\subset\mathfrak{A}'\subset\mathfrak{A}.$$

Examples functions from the set $\overline{F}$:

$\psi_1(t)=t^{-r}$, $r>0$; $\psi_2=t^{-r}\ln^\alpha(t+c)$, $r>0$, $\alpha\in\mathbb{R}$, $c\geq 1$; $\psi_3(t)=exp(-\alpha t^r)$, $\alpha>1$, $r>0$.

The main result of our work is

Theorem (Silin, 2010). *Let* $\psi_1, \psi_2 \in \overline{F}$ *and there exist constants* K_1 *and* K_2 *for such that*

$$0 < K_1 \le \frac{\eta(\psi_1; t) - t}{\eta(\psi_2; t) - t} \le K_2 < \infty, \quad t \ge 1; \tag{9}$$

α_i, δ_i *are quantities uniformly bounded in* σ. *Then for real numbers* $\sigma > h \ge 1$ *as* $\sigma \to \infty$ *and* $(\eta(\psi; \sigma) - \sigma)^{-1} > \frac{\pi}{h}$

$$\Sigma^{\alpha,\delta}_{\sigma,h,m}(\widehat{C}^{\overline{\psi}}_{\infty}) = \frac{4|\overline{\psi}(\sigma)|}{\pi^2} R_m \left| \ln \frac{\eta(\sigma) - \sigma}{h} \right| + O(1)|\overline{\psi}(\sigma - h)|, \tag{10}$$

$$\Sigma^{\alpha,\delta}_{\sigma,h,m}(\widehat{C}^{\overline{\psi}} H_\omega) = \frac{2\theta_\omega}{\pi^2} |\overline{\psi}(\sigma)| R_m \left| \ln \frac{\eta(\sigma) - \sigma}{h} \right| \int\limits_0^{\frac{\pi}{2}} \omega \left(\frac{2t}{\sigma} \right) \sin t \; dt +$$

$$+ O(1)|\overline{\psi}(\sigma - h)| \omega \left(\frac{1}{\sigma - h} \right), \tag{11}$$

where

$$R_m = \sqrt{A_m^2 + B_m^2}, \quad A_m = \sum_{i=1}^{m} \alpha_i \cos(\sigma \delta_i + \gamma),$$

$$B_m = \sum_{i=1}^{m} \alpha_i \sin(\sigma \delta_i + \gamma), \quad \gamma = \operatorname{arctg} \frac{\psi_2(\sigma)}{\psi_1(\sigma)},$$

$\eta(\sigma)$ *is* $\eta(\psi_1; \sigma)$ *or* $\eta(\psi_2; \sigma)$, $\theta_\omega \in [2/3, 1]$, *and* $\theta_\omega = 1$, *if* $\omega(t)$ *is a convex modulus of continuity, and* $O(1)$ *are quantity uniformly bounded in* σ *and* h.

If $(\eta(\psi; \sigma) - \sigma)^{-1} < \frac{\pi}{h}$, *then formulas (10) - (11) take place at condition that*

$$\delta_i = O((\eta(\psi; \sigma) - \sigma)^{-1}), \quad i = \overline{1, m}.$$

***Remark* 1.** The given theorem in a case $h=1$, $\psi_1(\sigma)=\psi(\sigma)\cos\frac{\beta\pi}{2}$, $\psi_2(\sigma)=\psi(\sigma)\sin\frac{\beta\pi}{2}$, $\beta\in\mathbb{R}$ and

$$\lambda^*_{\sigma,1}(t)=\begin{cases}1, & 0\le t\le\sigma-1,\\ 1-\frac{(t-\sigma+1)\psi(\sigma)}{\psi(t)}, & \sigma-1\le t\le\sigma,\\ 0, & t\ge\sigma\end{cases}$$

were obtained by Drozd in [7].

Theorem A (Drozd). *Let $\psi\in\overline{F}$, α_i, δ_i are quantities uniformly bounded in σ. Then, if $\frac{t}{\eta(\psi;t)-t}\ge t$, that for any real numbers $\sigma\ge1$*

$$\Sigma^{\alpha,\delta}_{\sigma,h,m}(\widehat{C}^{\overline{\psi}}_{\beta,\infty})=\frac{4\psi(\sigma)}{\pi^2}R_m\left|\ln\eta(\sigma)-\sigma\right|+O(1)\psi(\sigma); \tag{12}$$

$$\Sigma^{\alpha,\delta}_{\sigma,h,m}(\widehat{C}^{\overline{\psi}}_{\beta}H_\omega)=\frac{2\theta_\omega}{\pi^2}|\overline{\psi}(\sigma)|R_m\left|\ln\eta(\sigma)-\sigma\right|\int\limits_0^{\frac{\pi}{2}}\omega\left(\frac{2t}{\sigma}\right)\sin t\;dt+$$

$$+O(1)\psi(\sigma)\omega\left(\frac{1}{\sigma}\right), \tag{13}$$

where

$$R_m=\sqrt{A_m^2+B_m^2},\quad A_m=\sum_{i=1}^m\alpha_i\cos(\beta\pi/2-\sigma\delta_i),$$

$$B_m=\sum_{i=1}^m\alpha_i\sin(\beta\pi/2-\sigma\delta_i),$$

$\theta_\omega\in[2/3,1]$, *and* $\theta_\omega=1$, *if* $\omega(t)$ *is a convex modulus of continuity, and* $O(1)$ *quantities uniformly bounded in* σ,β.

If $\frac{t}{\eta(\psi;t)-t}\le t$, *then formulas (12) - (13) take place if*

$$\delta_i=O((\eta(\psi;\sigma)-\sigma)^{-1}),\quad i=\overline{1,m}.$$

We investigate possibility of justice of equality $R_m = 0$. Let in the beginning $m = 2$. Then

$$R_m = R_m(\alpha, \delta) = 0 \Leftrightarrow \begin{cases} \alpha_1 \cos(\sigma\delta_1 + \gamma) + \alpha_2 \cos(\sigma\delta_2 + \gamma) = 0, \\ \alpha_1 \sin(\sigma\delta_1 + \gamma) + \alpha_2 \sin(\sigma\delta_2 + \gamma) = 0, \end{cases}$$

That is equivalent to a system

$$\begin{cases} \delta_1 + \delta_2 = (\pi k)/\sigma, \\ \alpha_1 = (-1)^{k+1}\alpha_2, \end{cases} \quad k \in \mathbb{Z}. \tag{14}$$

Thus, in a case $m = 2$,

$$||f(x) + (-1)^{k+1} f(x + \pi k/\sigma) - V_{\sigma,h}\left(f(x) + (-1)^{k+1} f(x + \pi k/\sigma)\right)||_C \leq$$

$$\leq \begin{cases} \mathcal{K}|\overline{\psi}(\sigma - h)|\omega\left(\frac{1}{\sigma-h}\right), f \in \widehat{C}^{\overline{\psi}} H_\omega, \\ \mathcal{K}|\overline{\psi}(\sigma - h)|, f \in \widehat{C}^{\overline{\psi}}_\infty. \end{cases}$$

In the case $\psi_j \in \overline{F} \setminus (\mathfrak{A}_C \cup \mathfrak{A}^+_\infty)$, $j = 1, 2$ the relation

$$\pi k/\sigma = O((\eta(\psi; \sigma) - \sigma)^{-1})$$

is not true. Therefore, the conditions (14) is not satisfied.

Corollary. *Let $\psi_j \in \mathfrak{A}_C \cup \mathfrak{A}^+_\infty$, $j = 1, 2$ and such that satisfied the condition (9). Then for $\sigma > h \geq 1$ as $\sigma \to \infty$, $m = 2$ and quantities α_1, α_2, δ_1, δ_2 satisfied a condition (14)*

$$\Sigma^{\alpha,\delta}_{\sigma,h,m}(\widehat{C}^{\overline{\psi}}_\infty) = O(1)|\overline{\psi}(\sigma - h)|, \tag{15}$$

$$\Sigma^{\alpha,\delta}_{\sigma,h,m}(\widehat{C}^{\overline{\psi}} H_\omega) = O(1)|\overline{\psi}(\sigma - h)|\ \omega\left(1/(\sigma - h)\right), \tag{16}$$

where $O(1)$ are quantity uniformly bounded in σ, h.

If at $m \geq 3$ of coordinate α_i the are any vector $(\alpha_1, \ldots, \alpha_m)$ satisfy to one of conditions:

1) $\sum\limits_{i=1}^{m}(-1)^{k_i}\alpha_i=0$, *if the sum* $\delta_s+\delta_l=\frac{\pi n}{\sigma}$, $n\in\mathbb{N}$, *for all* $l,s\le m$, *thus* k_i *integer number, which it is defined from equiv*

$\delta_i=\delta_0+\pi k_i/\sigma$, *where* δ_0 *some fixed number;*

2) $\alpha_l=\sin^{-1}(\delta_s+\delta_l)\sigma\sum\limits_{i\neq l,s}\alpha_i\sin(\delta_i+\delta_s)\sigma$,

$\alpha_s=\sin^{-1}(\delta_s+\delta_l)\sigma\sum\limits_{i\neq l,s}\alpha_i\sin(\delta_i+\delta_l)\sigma$, *if for some* l *and* s

$\delta_s+\delta_l\neq\pi n/\sigma$, $n\in\mathbb{N}$,

then formulas (15) and (16) is true as $\sigma\to\infty$.

Remark 2. Suppose that $\psi_j\in\overline{F},\ j=1,2$ quantities $h=h(\sigma)$ are chosen so that $\sigma-h\in[\eta^{-1}(\psi_i;\sigma),\sigma]$, $i=1,2$. Then the following equalities hold as $\sigma\to\infty$

$$\psi(\sigma-h)=O(1)\psi(\sigma)$$

and

$$\omega\left(1/(\sigma-h)\right)=O(1)\omega\left(1/\sigma\right).$$

Therefore main members of the formulas (10) - (11) and (15) - (16) are $O(1)|\overline{\psi}(\sigma)|$ and $O(1)|\overline{\psi}(\sigma)|\omega\left(1/\sigma\right)$ accordingly.

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Examples.

1. $\psi_1(t) = t^{-r} \in \mathfrak{A}_C$, $r > 0$, $a(\sigma) = \frac{1}{\sigma(2^{1/r}-1)}$. $a(\sigma) < \pi/h \Leftrightarrow h < \pi\sigma(2^{1/r} - 1)$, that takes place as $\sigma > h$, i.e. always.

2. $\psi_2(t) = e^{-t} \in \overline{F}$, then $a(\sigma) = 1/\ln 2 \Rightarrow a(\sigma) > \pi/h$ at enough big h.

3. $\psi_3(t) = e^{-\ln 2t^2}$, then $a(\sigma) = \frac{1}{\sigma^2(\sqrt{1+\sigma^2}+1)} < \pi/h$.

4. $\psi_4(t) = e^{-\ln 2\sqrt{t}} \in \overline{F}_\infty$, $\psi_4 \notin \mathfrak{A}_C$, $a(\sigma) = \frac{1}{1+2\sqrt{\sigma}} > \pi/h \Leftrightarrow h > \pi(1 + 2\sqrt{\sigma})$. (For example, $\sigma = n^2$, $h = n^2 - \pi$, at $n > 2$ the inequality is fair.)

5. In sets $\mathfrak{A}_\infty \backslash \overline{F}$ and $\overline{F} \backslash (\mathfrak{A}_C \cup \mathfrak{A}_\infty^+)$ there are functions, for which $\eta(t) - t < 2$. Hence $a(\sigma) > 1/2$. Therefore at $\sigma \to \infty$ an inequality $\pi k/\sigma \leq Ma(\sigma)$ it is not executed.