Best m-term approximation and tensor products of Nikol'skij-Besov spaces

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1. Introduction

- Nikol'skij-Besov spaces and approximation theory
- Multivariate problems: $s > 0$, $1 \le p, q \le \infty$,

$$a_m(I, B^s_{p,q}([0,1]^d), L_p([0,1]^d)) \asymp m^{-s/d}\,.$$

Example: $s = 1$, $\varepsilon = 1/100 \Longrightarrow m \ge 100^d$.

- Function spaces with a dominating mixed derivative: Babenko, Bahkvalov, Smolyak, Mityagin, Korobov, Nikol'skij, Amanov, ..., Lizorkin, ... , Temlyakov, Dinh Dung, ..., Wasilkowski, Wozniakowski, ..., Kashin, Temirgaliev, Novak, Ritter, Schmeißer, Triebel, Oswald, Tikhomirov, Galeev, Magaril-Ilyaev, DeVore, Konyagin,..., Zenger, Griebel, Bungartz, Bazarkhanov, Vybiral, Ullrich, Hansen.

2.1 Some definitions

Tensor products of functions: $f, g : \mathbb{R} \to \mathbb{C}$,

$$(f \otimes g)(x, y) := f(x)\, g(y)$$

Tensor products of sequences: $a = (a_i)_{i=1}^\infty$, b=$(b_i)_{i=1}^\infty$.

$$a \otimes b := (a_i\, b_j)_{i,j=1}^\infty$$

X, Y quasi-normed spaces.

$$X \otimes Y := \Big\{ \sum_{i=1}^n f_i \otimes g_i : \quad f_i \in X\,,\ g_i \in Y,\ n \in \mathbb{N} \Big\}.$$

$$X \otimes_\alpha Y := \overline{X \otimes Y}^{\alpha}\,.$$

The p-nuclear norm

$\underline{1 \le p < \infty}$:

$$\alpha_p(h, X, Y) \;:=\; \inf \Bigg\{ \Big(\sum_{i=1}^{n} \|f_i|X\|^p \Big)^{1/p} \times \sup \Big\{ \Big\| \sum_{i=1}^{n} \lambda_i g_i \Big| Y \Big\| : \Big(\sum_{i=1}^{n} |\lambda_i|^p \Big)^{1/p} \le 1 \Big\} \Bigg\}$$

where the infimum is taken over all representations of h s.t.

$$h = \sum_{i=1}^{n} f_i \otimes g_i \,, \quad f_i \in X,\; g_i \in Y \,.$$

Lemma 1. (a) $\ell_p(\mathbb{N}) \otimes_{\alpha_p} \ell_p(\mathbb{N}) = \ell_p(\mathbb{N} \times \mathbb{N})$.
(b) $L_p(\Omega_1) \otimes_{\alpha_p} L_p(\Omega_2) = L_p(\Omega_1 \times \Omega_2)$.

$\underline{0 < p < 1}$: (Grothendieck, 1956)

$$\alpha_p(h, X, Y) := \inf \left\{ \Big(\sum_{i=1}^{n} \|f_i|X\|^p \|g_i|Y\|^p \Big)^{1/p} \right\}$$

where the infimum is taken over all representations of h s.t.

$$h = \sum_{i=1}^{n} f_i \otimes g_i\,, \quad f_i \in X,\ g_i \in Y\,.$$

Lemma 2. Let $0 < p < 1$. Then $\ell_p(\mathbb{N}) \otimes_{\alpha_p} \ell_p(\mathbb{N}) = \ell_p(\mathbb{N} \times \mathbb{N})$.

Tensor products of operators

$T_i: \ Y_i \to X_i$, $i = 1, 2$, linear.

$$(T_1 \otimes T_2)(f \otimes g) := (T_1 f) \otimes (T_2 g)\,, \qquad f \in Y_1,\ g \in Y_2\,.$$

$$(T_1 \otimes T_2)\Big(\sum_{i=1}^{n} f_i \otimes g_i\Big) := \sum_{i=1}^{n} (T_1 f_i) \otimes (T_2 g_i)$$

A quasi-norm α is called a crossnorm on $X \otimes Y$ if

$$\alpha(f \otimes g, X, Y) = \|f|X\|\,\|g|Y\|\,, \qquad f \in X,\ g \in Y\,.$$

A quasi-norm α is called uniform on $X \otimes Y$ if

$$\alpha\Big((T_1 \otimes T_2)h,\, X_1, X_2\Big) \le \|\, T_1\,|\mathcal{L}(Y_1, X_1)\|\,\|\, T_2\,|\mathcal{L}(Y_2, X_2)\|\,\alpha(h, Y_1, Y_2)$$

for all $h \in Y_1 \otimes Y_2$.

2.2 Tensor products of Sobolev spaces

$f, g \in C^1(\mathbb{R})$:

$$f \otimes g,\ f' \otimes g,\ f \otimes g', f' \otimes g' \in C(\mathbb{R})\,.$$

$C^m(\mathbb{R}^2)$: $\quad D^\alpha h \in C(\mathbb{R}^2)$ for all $|\alpha| \le m$.

$$C^2(\mathbb{R}^2) \subset C^1(\mathbb{R}) \otimes C^1(\mathbb{R}) \subset C^1(\mathbb{R}^2)$$

Sobolev spaces of dominating mixed smoothness

Let $d \ge 2$, $1 < p < \infty$ and $r \in \mathbb{N}$. Then $S^r_pW(\mathbb{R}^d)$ is the collection of all functions in $h \in L_p(\mathbb{R}^d)$ s.t.

$$\| h \,|S^r_pW(\mathbb{R}^d)\| := \sum_{\alpha \le (r, \ldots, r)} \| D^\alpha h | L_p(\mathbb{R}^d)\| < \infty\,.$$

- Nikol'skij 1962/63.

Derivative of the highest order: $D^{(r,\dots,r)}h$.

$$W_p^{rd}(\mathbb{R}^d) \hookrightarrow S_p^r W(\mathbb{R}^d) \hookrightarrow W_p^r(\mathbb{R}^d)\,.$$

Proposition 1. (S./Ullrich (2008))
Let $d \geq 1$, $1 < p < \infty$ and $r \in \mathbb{N}$. Then

$$S_p^r W(I^2) = W_p^r(I) \otimes_{\alpha_p} W_p^r(I)$$

and

$$S_p^r W(I^{d+1}) = S_p^r W(I^d) \otimes_{\alpha_p} W_p^r(I) = W_p^r(I) \otimes_{\alpha_p} S_p^r W(I^d)\,,$$

where I is either $\mathbb{R}$ or $I = [0, 1]$.

Remark. $S_2^r W(I^d)$ is also denoted as $H^r_{mix}(I^d)$.

2.3 Tensor products of Nikol'skij-Besov spaces

Proposition 2. (S./Ullrich (2009))
Let $d \geq 1$, $0 < p < \infty$ and $r \in \mathbb{R}$. Then

$$S^r_{p,p}B(I^2) = B^r_{p,p}(I) \otimes_{\alpha_p} B^r_{p,p}(I)$$

and

$$S^r_{p,p}B(I^{d+1}) = S^r_{p,p}B(I^d) \otimes_{\alpha_p} B^r_{p,p}(I) = B^r_{p,p}(I) \otimes_{\alpha_p} S^r_{p,p}B(I^d)\,,$$

where I is either $\mathbb{R}$ or $I = [0,1]$.

Sobolev embedding: $0 < p_0 < p_1 < \infty$

$$S^{r_0}_{p_0,p_0}B(I^d) \hookrightarrow L_{p_1}(I^d) \qquad \Longleftrightarrow \qquad r_0 \geq \frac{1}{p_0} - \frac{1}{p_1}\,.$$

Tensor product Besov (Sobolev) spaces are much smaller than the isotropic spaces !

Approximation numbers (linear widths)

$T : X \to Y$, linear,

$$a_n(T, X, Y) := \inf\{\| T - L \,|\mathcal{L}(X, Y)\| \,:\, L \text{ linear}, \quad \mathrm{rank}\, L \le n\}$$

$$a_0(T, X, Y) := \| T \,|\mathcal{L}(X, Y)\| \,, \qquad a_n \downarrow \,.$$

Approximation numbers – tensor product spaces

Babenko (1960), Mityagin (1962), Temlyakov (1993), Galeev (2001), Romanyuk (1991-2008).

Let $r > 0$ and $1 < p < \infty$. Then

$$a_m \quad (I, S^r_{p,p}B((0,1)^d), L_p((0,1)^d))$$

$$\asymp \quad m^{-r}\,(\log m)^{(d-1)r} \begin{cases} (\log m)^{(d-1)(\frac{1}{2}-\frac{1}{p})} & \text{if} \quad 2 \le p < \infty\,, \\ 1 & \text{if} \quad 1 < p \le 2\,, \end{cases}$$

holds.

An example from quantum chemistry

The electronic Hamilton operator (associated to the electronic Schrödinger equation) is given by

$$H := -\frac{1}{2}\sum_{i=1}^{N}\Delta_i - \sum_{i=1}^{N}\frac{1}{2}\sum_{\nu=1}^{K}\frac{Z}{|x_i - a_\nu|} + \frac{1}{2}\sum_{\substack{i\neq j\\ 1\le i,j\le N}}\frac{1}{|x_i - x_j|}$$

$x_i = (x_i^1, x_i^2, x_i^3)$ - coordinates of the i-th electron
Δ_i - Laplace operator with respect to x_i

$$H:\ W_2^1(\mathbb{R}^{3N}) \to W_2^{-1}(\mathbb{R}^{3N})\,.$$

Problem: approximation of the eigenfunctions of H.

Known properties of the eigenfunctions $u \in H^1(\mathbb{R}^{3N})$
(Yserentant 2003/2004/2007, Lect. Notes in Math. **2000**)

(i) There exists $\delta > 0$ s.t.

$$e^{\delta\,|x|}\,(|u(x)| + |\nabla(x)|) \in L_2(\mathbb{R}^{3N})\,.$$

(ii) $\quad u \in W_2^1(\mathbb{R}^3) \otimes_{\alpha_2} \ldots \otimes_{\alpha_2} W_2^1(\mathbb{R}^3).$

Remark:

(a) $S_2^1 W(\mathbb{R}^3 \times \ldots \times \mathbb{R}^3)$ is smaller than $W_2^1(\mathbb{R}^{3N})$.

(b) Hansen (2010): $S^r_{p,q}B(\mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_N})$ and best m-term approximation

$$a \in b^r_p : \qquad \| \, a \, | b^r_p \| := \Big(\sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} 2^{j(r+\frac{1}{2}-\frac{1}{p})p} \, |a_{j,k}|^p \Big)^{1/p} < \infty \, .$$

$$s^r_p b := b^r_p \, \otimes_{\alpha_p} \ldots \otimes_{\alpha_p} \, b^r_p \, .$$

Let

$$J_i : \, B^r_{p,p}(\mathbb{R}) \longrightarrow b^r_p, \qquad i = 1, \ldots , d \, ,$$

be isomorphisms. Then

$$J := J_1 \otimes \ldots \otimes J_d : \, S^r_{p,p}B(\mathbb{R}^d) \longrightarrow s^r_p b$$

is an isomorphism as well.

3.1 Characterizations of tensor product Nikol'skij-Besov spaces

Based on Lemma 1/2:

$$\| \, a \, | s^r_p \| := \Big(\sum_{j \in \mathbb{N}_0^d} 2^{|j|_1 (r + \frac{1}{2} - \frac{1}{p}) p} \sum_{k \in \mathbb{Z}^d} |a_{j,k}|^p \Big)^{1/p} < \infty \, .$$

Tensor product wavelet systems: $\vec{\Phi} := \Phi \otimes \ldots \otimes \Phi$

$$\Psi_{j,k}(x) := \psi_{j_1,k_1}(x_1) \cdot \, \ldots \cdot \psi_{j_d,k_d}(x_d)$$

$$J \otimes \ldots \otimes J \; : \quad f \mapsto (\langle f, \Psi_{j,k} \rangle)_{j,k}$$

Proposition 3. (Vybiral (2003)).
Let $r \in \mathbb{R}$ and $0 < p < \infty$. Let $\vec{\Phi}$ denote the Daubechies tensor product wavelet system of sufficiently high order. Then

$$\| f \,| S^r_{p,p} B(\mathbb{R}^d)\| \asymp \| (\langle f, \Psi_{j,k} \rangle)_{j,k} \,| s^r_p b\| \,.$$

Observation:

- $s^0_2 b = \ell_2(\mathbb{N}^d_0 \times \mathbb{Z}^d)$;
- $s_p^{\frac{1}{p}-\frac{1}{2}} b = \ell_p(\mathbb{N}^d_0 \times \mathbb{Z}^d), \quad 0 < p < \infty.$

Approximation schemes

An approximation scheme (X, A_n) is a quasi-Banach space X together with a sequence of subsets $A_n \subset X$ s.t.

- $A_1 \subset A_2 \subset \ldots \subset X$.
- $\lambda A_n \subset A_n$ for all scalars λ and all n.
- $A_m + A_n \subset A_{m+n}$ for all n, m.

It is convenient to define $A_0 = \{0\}$.

Let $f \in X$. The associated approximation numbers are defined as

$$\alpha_n(f, X) := \inf \Big\{ \| f - g \,|X\| : \; g \in A_{n-1} \Big\} , \qquad n \in \mathbb{N} .$$

Approximation spaces

Definition. Let $0 < s < \infty$ and $0 < q \le \infty$. Then the approximation space $\mathcal{A}^s_q := (X, A_n)^s_q$ consists of all elements $f \in X$ s.t.

$$\| f \,|\mathcal{A}^s_q\| := \begin{cases} \Big(\sum\limits_{n=1}^{\infty} n^{sq-1}\, \alpha_n(f,X)^q \Big)^{1/q} & 0 < q < \infty\,, \\ \sup_{n=1,2,\ldots}\; n^s\, \alpha_n(f,X) & q = \infty\,, \end{cases}$$

is finite.

Lemma 3. $\mathcal{A}^s_q$ is a quasi-Banach space. The scale is monotone in q, i.e.

$$\mathcal{A}^s_{q_0} \hookrightarrow \mathcal{A}^s_{q_1} \hookrightarrow \mathcal{A}^s_{\infty}\,, \qquad q_0 \le q_1\,.$$

Example

Let $\mathcal{D} = \{\psi_1, \psi_2, \ldots\}$ be a countable subset of X. Then we may take

$$A_n := \Big\{ \sum_{i \in \Lambda} a_i \, \psi_i : \ \Lambda \subset \mathbb{N}, \ |\Lambda| < n \, , a_i \in \mathbb{C} \Big\} \, .$$

$$\sigma_m(f, \mathcal{D})_X := \inf \Big\{ \| \, f - g \, |X\| : \ g \in A_{m+1} \Big\} \, , \qquad n \in \mathbb{N} \, .$$

- $\mathcal{A}^s_q = \mathcal{A}^s_q(X, \mathcal{D})$.

I - infinite countable set;
$\mathcal{B}$ - the canonical basis in $\ell_2(I)$.

Lemma 4. (Pietsch 1981). Let $0 < p_1, q \le \infty$. Let I be a countable infinite set. Then $a \in \ell_{p_1}(I)$ belongs to the approximation space $\mathcal{A}_q^s(\ell_{p_1}(I), \mathcal{B})$, if and only if $a \in \ell_{p_0,q}(I)$, where $1/p_0 := s + 1/p_1$. Furthermore,

$$\big\| \, a \, \big| \mathcal{A}_q^s \big(\ell_{p_1}(I), \mathcal{B}\big) \big\| \asymp \big\| \, a \, \big| \ell_{p_0,q}(I) \big\| \, , \qquad (1)$$

where the constants of equivalence do not depend on I.

$q = p_0$: $$\mathcal{A}_{p_0}^{\frac{1}{p_0} - \frac{1}{p_1}} \big(\ell_{p_1}(I), \mathcal{B}\big) = \ell_{p_0}(I).$$

$$s_p^{\frac{1}{p}-\frac{1}{2}} b = \ell_p(\mathbb{N}_0^d \times \mathbb{Z}^d).$$

Lemma 4 yields in case $p_0 < p_1$

$$\mathcal{A}_{p_0}^{\frac{1}{p_0}-\frac{1}{p_1}}(s_{p_1}^{\frac{1}{p_1}-\frac{1}{2}} b, \mathcal{B}) = \mathcal{A}_{p_0}^{\frac{1}{p_0}-\frac{1}{p_1}}(\ell_{p_1}, \mathcal{B}) = \ell_{p_0} = s_{p_0}^{\frac{1}{p_0}-\frac{1}{2}} b$$

$$r \in \mathbb{R}: \qquad \mathcal{A}_{p_0}^{\frac{1}{p_0}-\frac{1}{p_1}}(s_{p_1}^{r+\frac{1}{p_1}-\frac{1}{2}} b, \mathcal{B}) = s_{p_0}^{r+\frac{1}{p_0}-\frac{1}{2}} b$$

$$r = \frac{1}{2} - \frac{1}{p_1}: \qquad \mathcal{A}_{p_0}^{\frac{1}{p_0}-\frac{1}{p_1}}(s_{p_1}^{0} b, \mathcal{B}) = s_{p_0}^{\frac{1}{p_0}-\frac{1}{p_1}} b$$

Special case: $p_1 = 2$.
$\Phi = \{\psi_1, \psi_2, \ldots\}$ - wavelet basis of tensor product type.

Corollary. Let $0 < p_0 < 2$ and define $r := \frac{1}{p_0} - \frac{1}{2}$.

(i) The function $f \in L_2(\mathbb{R}^d)$ belongs to the approximation space $\mathcal{A}^r_{p_0}(L_2(\mathbb{R}^d), \vec{\Phi})$, if and only if the sequence $(\langle f, \Psi_{j,k}\rangle)_{j,k}$ of its Fourier coefficients belongs to ℓ_{p_0}.

(ii) We also have

$$\sigma_m\Big(S^{\frac{1}{p_0}-\frac{1}{2}}_{p_0,p_0}B(\mathbb{R}^d), L_2(\mathbb{R}^d), \vec{\Phi}\Big) \asymp m^{-\frac{1}{p_0}+\frac{1}{2}}$$

Remarks: (i) $\mathcal{A}^s_q \hookrightarrow \mathcal{A}^s_\infty$.
(ii) Nitsche (2006).

Theorem 1. (S./Hansen (2010)).

We assume $\max(1, p_0) < p_1 < \infty$, and $r_0 := \frac{1}{p_0} - \frac{1}{p_1}$.

In case of tensor product Besov spaces we have

$$\sigma_m\Big(S^{r_0}_{p_0,p_0}B(\mathbb{R}^d), L_{p_1}(\mathbb{R}^d), \vec{\Phi}\Big) \asymp m^{-r_0}\,(\log m)^{(d-1)(r_0-\frac{1}{p_0}+\frac{1}{2})_+}$$

for all $m \geq 2$.

Remark. Dominating mixed counterpart of the famous result of DeVore, Jawerth and Popov (1992).

3.4 Compact embeddings

Vybiral (2003): $1 \le p_1 \le \infty$.

$$S^{r_0}_{p_0,p_0}B([0,1]^d) \hookrightarrow\hookrightarrow L_{p_1}([0,1]^d) \quad \Longleftrightarrow \quad r_0 > \max\Big(0, \frac{1}{p_0} - \frac{1}{p_1}\Big).$$

Theorem 2. (S./Hansen (2010)).
We assume $0 < p_0 < \infty$, $1 < p_1 < \infty$, and $r_0 > \max(0, \frac{1}{p_0} - \frac{1}{p_1})$.
In case of tensor product Besov spaces we have

$$\sigma_m\Big(S^{r_0}_{p_0,p_0}B([0,1]^d), L_{p_1}([0,1]^d), \vec{\Phi}\Big) \asymp m^{-r_0} (\log m)^{(d-1)(r_0 - \frac{1}{p_0} + \frac{1}{2})_+}$$

for all $m \ge 2$.

• Oswald (1999): Haar system, $p_1 = p_2 = 2$, $0 < r < 1/2$;

• Temlyakov (2000): periodic Sobolev and Nikol'skij-Besov spaces; Let $1 < p_0, p_1 < \infty$ and

$$r_0 > \begin{cases} \left(\frac{1}{p_0} - \frac{1}{p_1}\right)_+ & \text{falls} \quad p_1 \geq 2\,, \\ \frac{1}{p_0}\left(2\max\left(\frac{1}{p_0}, \frac{1}{p_1}\right) - 1\right) & \text{falls} \quad 1 < p_1 < 2\,. \end{cases}$$

Then

$$\sigma_m\left(\widetilde{S}^{r_0}_{p_0,\infty}B(\mathbb{T}^n), L_{p_1}(\mathbb{T}^n), U^d\right) \asymp m^{-t}\,(\log m)^{(d-1)(t+\frac{1}{2})}$$

holds for all $m \geq 2$.
U^d - an appropriate orthonomal basis in $L_2(\mathbb{T}^n)$, which consists of trigonometric polynomials.

Theorem 3. (S./Hansen (2010)).

Let $0 < p_0, p_1 q_0, q_1 \leq \infty$. We suppose

$$r_0 - r_1 > \max\left(0, \frac{1}{p_0} - \frac{1}{p_1}\right).$$

Then we have

$$\begin{aligned}\sigma_m & \left(S^{r_0}_{p_0,q_0}B([0,1]^d), S^{r_1}_{p_1,q_1}B([0,1]^d), \vec{\Phi}\right) \\ & \asymp \quad m^{-r_0+r_1}(\log m)^{(d-1)\left(r_0-r_1-\frac{1}{q_0}+\frac{1}{q_1}\right)_+}, \quad m \geq 2.\end{aligned}$$

- Dinh Dung (2000/2001).

4. A final remark

$$r_0 > \max\left(0, \frac{1}{p_0} - \frac{1}{p_1}\right):$$

$$\sigma_m\left(S^{r_0}_{p_0,p_0}B([0,1]^d), L_{p_1}([0,1]^d), \vec{\Phi}\right) \asymp m^{-r_0}\,(\log m)^{(d-1)(r_0-\frac{1}{p_0}+\frac{1}{2})_+}$$

Take $r_0 = 1,\ d = 11,\ p_0 = p_1 = 2$:

$$\sigma_m\left(S^1_2 W([0,1]^d), L_2([0,1]^d), \vec{\Phi}\right) \asymp m^{-1}\,(\log m)^{10}$$

With $m = 2^{10}$ we get

$$m^{-1}\,(\log m)^{10} = 2^{-10}\,10^{10} = 5^{10}\,.$$

With $m = 2^{20}$ we get

$$m^{-1}\,(\log m)^{10} = 2^{-20}\,20^{10} = 5^{10}\,.$$