Optimal embeddings of spaces of generalized smoothness in the critical case

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[1] joint work with Susana Moura and Júlio Neves

Outline

Function spaces of generalized smoothness

- (Generalized) Besov spaces
- (Generalized) Hölder spaces

Main results

- The critical case
- Optimal embeddings

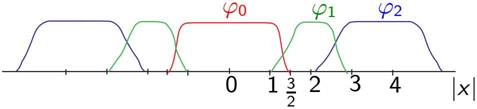
Applications

- Approximation numbers

Besov spaces

$\{\varphi_k\}_{k=0}^{\infty} \ldots$ smooth dyadic resolution of unity, i.e.,

- $\displaystyle\sum_{k=0}^{\infty} \varphi_k = 1,$



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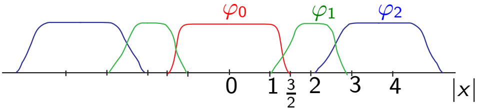

▶ Besov spaces: $0 < p, q \leq \infty,\ s \in \mathbb{R}$

$$\|f|B^s_{pq}\| = \left(\sum_{k=0}^{\infty} 2^{ksq} \|(\varphi_k \hat{f})^{\vee} | L_p\|^q \right)^{1/q}$$

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- Slowly varying function Ψ: positive, measurable function on $(0, 1]$ with

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$$\Psi(x) = (1+|\log x|)^a \left(1+\log\left(1+|\log x|\right)\right)^b, \quad x \in (0,1],\ a,b \in \mathbb{R},$$

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Definition

Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$, Ψ slowly varying function. The space $B^{(s,\Psi)}_{pq}$ consists of all $f \in S'$ such that

$$\|f|B^{(s,\Psi)}_{pq}\| = \left(\sum_{k=0}^{\infty} 2^{ksq} \Psi(2^{-k})^q \|(\varphi_k \hat{f})^\vee | L_p\|^q \right)^{1/q} < \infty.$$

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$C_B \ldots$ space of bounded, continuous functions

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$$\omega(f,t) := \sup_{|h| \leq t} \sup_{x \in \mathbb{R}^n} |f(x+h) - f(x)|, \quad t > 0$$

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$$\left(\int_0^1 \frac{1}{\lambda(t)^r} \frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}} = \infty \qquad \text{and} \qquad \left(\int_0^1 \frac{t^r}{\lambda(t)^r} \frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}} < \infty.$$

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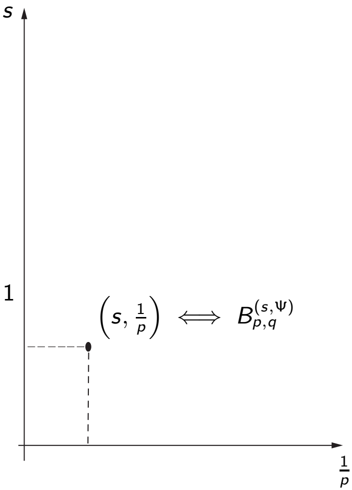
Definition

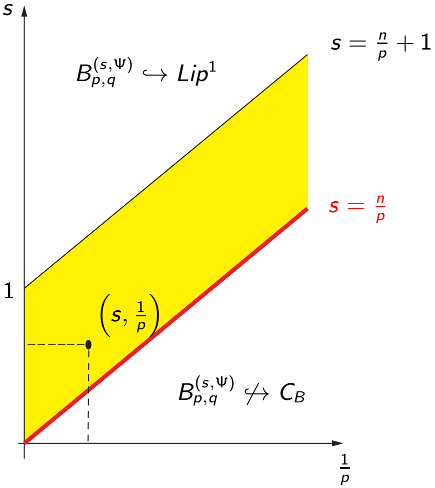
Let $0 < r \leq \infty$, $\mu \in \mathcal{L}_r$. The space $\Lambda^{\mu(\cdot)}_{\infty,r}$ consists of all $f \in C_B$ for which

$$||f|\Lambda^{\mu(\cdot)}_{\infty,r}|| := \|f|L_\infty\| + \left(\int_0^1 \left[\frac{\omega(f,t)}{\mu(t)} \right]^r \frac{\mathrm{d}t}{t} \right)^{\frac{1}{r}} < \infty$$

The critical case



Embedding in C_B?



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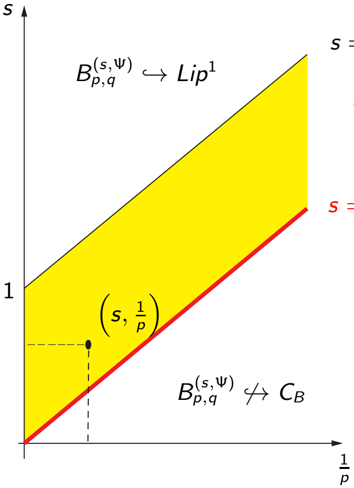

$s = \frac{n}{p} + 1$

$$B_{p,q}^{(\frac{n}{p}+1,\Psi)} \hookrightarrow Lip^1 \qquad \Longleftrightarrow \qquad \left(\Psi(2^{-j})^{-1}\right)_{j\in\mathbb{N}_0} \in \ell_{q'}$$

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Theorem (Moura, Neves, and S., 2010)

Let $0 < p \leq \infty$, $0 < q, r \leq \infty$, $\mu \in \mathcal{L}_r$, Ψ *a slowly varying function with*

$$\left(\Psi(2^{-j})^{-1}\right)_{j \in \mathbb{N}_0} \in \ell_{q'}.$$

(i) *If* $0 < q \leq r \leq \infty$, *then*

$$B^{(\frac{n}{p},\Psi)}_{p,q}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n),$$

if, and only if,

$$\sup_{N \geq 0} \left(\sum_{j=0}^{N} \int_{2^{-(j+1)}}^{2^{-j}} \mu(t)^{-r} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{r}} \left(\sum_{k=N}^{\infty} \Psi(2^{-k})^{-q'} \right)^{\frac{1}{q'}} < \infty.$$

(ii) *If* $0 < r < q \leq \infty$, *then ...*

Theorem (Moura, Neves, and S., 2010)

(ii) *If* $0 < r < q \leq \infty$, *then* $B^{(\frac{n}{p},\Psi)}_{p,q}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n)$ *if, and only if,*

$$\left\{\sum_{N=0}^{\infty}\left(\sum_{j=0}^{N}\int_{2^{-(j+1)}}^{2^{-j}}\mu(t)^{-r}\frac{\mathrm{d}t}{t}\right)^{\frac{u}{q}}\cdot\left(\int_{2^{-(N+1)}}^{2^{-N}}\mu(t)^{-r}\frac{\mathrm{d}t}{t}\right)\right.$$

$$\left.\cdot\left(\sum_{k=N}^{\infty}\Psi(2^{-k})^{-q'}\right)^{\frac{u}{q'}}\right\}^{\frac{1}{u}} < \infty$$

and

$$\left\{\sum_{N=0}^{\infty}\left(\sum_{j=N}^{\infty}2^{-jr}\int_{2^{-(j+1)}}^{2^{-j}}\mu(t)^{-r}\frac{\mathrm{d}t}{t}\right)^{\frac{u}{q}}\cdot 2^{-Nr}\left(\int_{2^{-(N+1)}}^{2^{-N}}\mu(t)^{-r}\frac{\mathrm{d}t}{t}\right)\right.$$

$$\left.\cdot\left(\sum_{k=0}^{N}2^{kq'}\Psi(2^{-k})^{-q'}\right)^{\frac{u}{q'}}\right\}^{\frac{1}{u}} < \infty,$$

where $\frac{1}{u} := \frac{1}{r} - \frac{1}{q}$.

Ideas of Proof:

- use equivalent characterization of Besov spaces via

  $\curvearrowright$ Peetre's maximal function

  $\curvearrowright$ atomic decompositions

- construct extremal functions $f_b \longrightarrow$ careful estimates for $w(f_b, 2^{-k})$
- Hardy inequalities for non-negative sequences

... long & technical calculations unavoidable in limiting case

Corollary (Moura, Neves, and S., 2010)

Let $1 < q \leq \infty$ *and define weights*

$$\lambda_{qr}(t) := \Psi(t)^{\frac{q'}{r}} \left(\int_0^t \Psi(s)^{-q'} \frac{\mathrm{d}s}{s} \right)^{\frac{1}{q'} + \frac{1}{r}}, \qquad t \in (0, 1].$$

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The embedding

$$B_{p,q}^{(\frac{n}{p},\Psi)} \hookrightarrow \Lambda_{\infty,r}^{\mu(\cdot)}, \qquad 1 < q \leq r \leq \infty,$$

...with $\mu = \lambda_{qr}$ *is sharp, i.e.,*

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...with $\mu = \lambda_{qq}$ *and* $r = q$ *is optimal, i.e.,*

$$\Lambda_{\infty,q}^{\lambda_{qq}(\cdot)} \hookrightarrow \Lambda_{\infty,r}^{\mu(\cdot)}.$$

Application: Approximation numbers

- Approximation numbers

$$a_k(\mathrm{id} : X(U) \longrightarrow C(U)) := \inf\big\{\|\mathrm{id} - L\| : L \in \mathcal{L}(X(U), C(U)),\ \mathrm{rank}\, L < k\big\}$$

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Theorem (Moura, Neves, and S., 2010)

Let $2 < p \leq \infty$, $0 < q \leq \infty$, *and assume*

$$\left(\Psi(2^{-j})^{-1}\right)_{j\in\mathbb{N}_0} \in \ell_{q'}.$$

For all $k \in \mathbb{N}$,

$$\Psi(k^{-\frac{1}{n}})^{-1} \lesssim a_k\left(\mathrm{id} : B_{pq}^{(\frac{n}{p},\Psi)}(U) \longrightarrow C(U)\right) \lesssim \left(\sum_{j=\left[\frac{\log k}{n}\right]}^{\infty} \Psi(2^{-j})^{-q'}\right)^{\frac{1}{q'}}$$

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Thank you!