

Equivalent norms and characterizations for vector-valued function spaces

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Guideline of the talk

Function Spaces $B_{p,q}^s(E)$, $F_{p,q}^s(E)$ of functions

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where E is a Banach space.

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$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m}. \quad \text{Not today!}$$

The spaces $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n, E)$

Let $\mathcal{S}(\mathbb{R}^n)$ be the set of the infinitely often differentiable functions $\varphi : \mathbb{R}^n \rightarrow \mathbb{C}$ for which the seminorms

$$\|\varphi\|_{K,L} := \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^{\frac{K}{2}} \sum_{|\alpha| \leq L} |D^\alpha \varphi(x)|$$

for $K, L \in \mathbb{N}_0$ are finite.

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for $K, L \in \mathbb{N}_0$ are finite.

We say that a linear map $f : \mathcal{S}(\mathbb{R}^n) \rightarrow E$ is an E -valued tempered distribution if there exist a constant $c > 0$ and $K, L \in \mathbb{N}_0$ such that

$$\|f(\varphi)\|_E \leq c \|\varphi\|_{K,L}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. The set of all this linear mappings equipped with the weak topology is denoted by $\mathcal{S}'(\mathbb{R}^n, E)$.

Bochner integral of vector-valued functions

Let $0 < p \leq \infty$. We define

$$L_p(E) := \left\{ f : \mathbb{R}^n \rightarrow E, f \text{ measurable}, \left\| \|f|E\| \right\|_{L_p(\mathbb{R}^n, \mathbb{C})} < \infty \right\}.$$

Warning! Unusual use of letter E in brackets.

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For

$$f = \sum_{m=1}^M a_m \cdot e_m$$

with $e_m \in E$ and $a_m \in L_1(\mathbb{C})$ we set

$$\int_{\mathbb{R}^n} f(x) \, dx := \sum_{m=1}^M e_m \cdot \int_{\mathbb{R}^n} a_m(x) \, dx \in E.$$

Bochner integral of vector-valued functions (ii)

It holds:

- The integral is independent of the representation.
- The subspace of functions of this type is dense in $L_1(E)$.
- We have the estimate

$$\left\| \int_{\mathbb{R}^n} f(x) \, dx|E \right\| \leq \left\| \|f|E\| |L_1(\mathbb{C}) \right\| = \|f|L_1(E)\|.$$

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⇒ There is a unique continuation: A continuous linear operator from $L_1(E)$ to E with norm 1 called Bochner integral.

⇒ For a linear functional $a : E \rightarrow \mathbb{C}$ one has

$$a \left(\int_{\mathbb{R}^n} f(x) \, dx \right) = \int_{\mathbb{R}^n} a(f(x)) \, dx.$$

Fourier transform of $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n, E)$

The Fourier transform $\hat{\varphi}$ of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is given by

$$\hat{\varphi}(\xi) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(x) e^{-ix\xi} dx,$$

analogously its inverse $\check{\varphi}$. We have $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$ iff $f \in \mathcal{S}(\mathbb{R}^n)$.

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The Fourier transform \hat{f} for $f \in \mathcal{S}'(\mathbb{R}^n, E)$ is given by

$$\hat{f}(\varphi) := f(\hat{\varphi}) \text{ for } \varphi \in \mathcal{S}(\mathbb{R}^n),$$

analogously its inverse \check{f} . We have $\hat{f} \in \mathcal{S}'(\mathbb{R}^n, E)$ iff $f \in \mathcal{S}'(\mathbb{R}^n, E)$.

Fourier transform for $L_1(E)$ and convolution

If $f \in L_p(E)$ for a $p \in [1, \infty]$, then we can think of f as an element of $\mathcal{S}'(\mathbb{R}^n, E)$ by defining

$$f(\varphi) := \int_{\mathbb{R}^n} \underbrace{f(x)}_{\in E} \underbrace{\varphi(x)}_{\in \mathbb{C}} dx.$$

Distributions of such an integral form are called regular.

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Distributions of such an integral form are called regular.

As in the scalar case we have: If $f \in L_1(E)$, then \hat{f} is a regular distribution, bounded and it holds

$$\hat{\varphi}(\xi) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(x) e^{-ix\xi} dx.$$

For $f \in \mathcal{S}'(\mathbb{R}^n, E)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we define the convolution of f and φ as the function

$$(f * \varphi)(x) := (2\pi)^{-\frac{n}{2}} \cdot f(\varphi(x - \cdot)) \text{ for all } x \in \mathbb{R}^n.$$

The definition of $B_{p,q}^s(E)$

Let $\{\varphi_j\}_{j=0}^\infty$ be a resolution of unity. Let $0 < p \leq \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. For $f \in \mathcal{S}'(\mathbb{R}^n, E)$ we define

$$\|f\|_{B_{p,q}^s(E)}^\varphi := \left(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \hat{f})^\vee\|_{L_p(E)}^q \right)^{\frac{1}{q}}$$

(modified in case $q = \infty$) and

$$B_{p,q}^{s,\varphi}(E) := \{f \in \mathcal{S}'(\mathbb{R}^n, E) : \|f\|_{B_{p,q}^s(E)}^\varphi < \infty\}.$$

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Then $(B_{p,q}^{s,\varphi}(E), \|\cdot\|_{B_{p,q}^s(E)}^\varphi)$ is a quasi-Banach space. It does not depend on the choice of the resolution of unity $\{\varphi_j\}_{j=0}^\infty$ in the sense of equivalent norms. So we denote it shortly by $B_{p,q}^s(E)$.

A deeper look at the resolution of unity

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- at first $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^n)$ to define a convolution
- Fourier multiplier theorems for functions whose Fourier transform have compact support:

$$\text{supp } \varphi_j \hat{f} \subset \text{supp } \varphi_j \subset \{2^{j-1} \leq |x| \leq 2^{j+1}\},$$

- the property $|D^\alpha \varphi_j(x)| \leq c_\alpha 2^{-j|\alpha|}$
- the unit property $\sum_{j=0}^{\infty} \varphi_j(x) = 1$.

A deeper look at the resolution of unity

The independency of the resolution of unity is shown by using

- at first $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^n)$ to define a convolution
- Fourier multiplier theorems for functions whose Fourier transform have compact support:

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- the property $|D^\alpha \varphi_j(x)| \leq c_\alpha 2^{-j|\alpha|}$
- the unit property $\sum_{j=0}^{\infty} \varphi_j(x) = 1$.

Question: Is there a possibility to weaken the conditions on φ_0, φ ?

Triebel's approach for the scalar case

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- not only discrete but also continuous norms

$$\left(\|(\varphi_0 \hat{f})^\vee\|_{L_p} + \sum_{j=1}^{\infty} 2^{jsq} \|(\varphi(2^{-j}\cdot) \hat{f})^\vee\|_{L_p}^q \right)^{\frac{1}{q}}$$
$$\Rightarrow \|(\varphi_0 \hat{f})^\vee\|_{L_p} + \left(\int_0^1 t^{-sq} \|(\varphi(t\cdot) \hat{f})^\vee\|_{L_p}^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

Rychkov's approach for the scalar case

V. S. Rychkov: On a Theorem of Bui, Paluszyński, and Taibleson.
Proc. Steklov Inst. Math. 227, 280, 1999

Let $S \in \mathbb{Z}$ with $S \geq \lfloor s \rfloor$, $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\varepsilon > 0$ such that

$$|\varphi_0(x)| > 0 \text{ for } \{|x| < 2\varepsilon\}, \quad |\varphi(x)| > 0 \text{ for } \left\{ \frac{\varepsilon}{2} < |x| < 2\varepsilon \right\},$$

$$D^\alpha \varphi(0) = 0 \text{ for } |\alpha| \leq S.$$

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Let $\varphi_j := \varphi(2^{-j}\cdot)$. Then

$$\|f\|_{B_{p,q}^s} := \|(\varphi_0 \hat{f})^\vee\|_{L_p} + \left(\sum_{j=1}^{\infty} 2^{jsq} \|(\varphi_j \hat{f})^\vee\|_{L_p}^q \right)^{\frac{1}{q}}$$

is an equivalent norm for $B_{p,q}^s$. It holds

$$B_{p,q}^s = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{B_{p,q}^s} < \infty \right\}.$$

Comparison of the two theorems/proofs

Advantages of the Triebel proof

- more general result (characterization by differences)
- no need of $\mathcal{S}(\mathbb{R}^n)$ functions
- continuous norms

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Advantages of the Rychkov proof

- much easier proof
- fewer conditions on φ , for $s < 0$ even no more conditions (this does not come out of Triebel's proof)
- gives characterizations for all s, p, q , not only equivalent norms

Rychkov's approach for discrete vector-valued norms

Theorem

Let $S \in \mathbb{Z}$ with $S \geq [s]$, $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\varepsilon > 0$ such that

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Let $\varphi_j := \varphi(2^{-j}\cdot)$. Then

$$\|f|B_{p,q}^s(E)\|^\varphi := \|(\varphi_0 \hat{f})^\vee|L_p(E)\| + \left(\sum_{j=1}^{\infty} 2^{jsq} \|(\varphi_j \hat{f})^\vee|L_p(E)\|^q \right)^{\frac{1}{q}}$$

is an equivalent norm for $B_{p,q}^s(E)$. It holds

$$B_{p,q}^s(E) = \{f \in \mathcal{S}'(\mathbb{R}^n, E) : \|f|B_{p,q}^s(E)\|^\varphi < \infty\}.$$

Rychkov's approach for **continuous** vector-valued norms

Theorem

Let $S \in \mathbb{Z}$ with $S \geq \lfloor s \rfloor$, $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\varepsilon > 0$ such that

$$|\varphi_0(x)| > 0 \text{ for } \{|x| < 2\varepsilon\}, \quad |\varphi(x)| > 0 \text{ for } \left\{\frac{\varepsilon}{2} < |x| < 2\varepsilon\right\},$$

$$D^\alpha \varphi(0) = 0 \text{ for } |\alpha| \leq S.$$

Then

$$\|f\|_{B_{p,q}^s(E)} \|' = \|(\varphi_0 \hat{f})^\vee\|_{L_p(E)} + \left(\int_0^1 t^{-sq} \|(\varphi(t \cdot) \hat{f})^\vee\|_{L_p(E)}^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

is an equivalent norm for $B_{p,q}^s(E)$. It holds

$$B_{p,q}^s(E) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n, E) : \|f\|_{B_{p,q}^s(E)} \|' < \infty \right\}.$$

Some improvement is possible

- One can replace $(\varphi_j \hat{f})^\vee$ by

$$(\varphi_j^* f)_a(x) := \sup_{y \in \mathbb{R}^n} \frac{\|(\varphi(2^{-j} \cdot) \hat{f})^\vee(x - y) \|_E\|}{(1 + 2^j |y|)^a},$$

where a must be bigger than $\frac{n}{p}$ and additionally bigger than $\frac{n}{q}$ for $F_{p,q}^s(E)$.

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- Let $d > 0$ be given. One can replace $\|(\varphi(t\cdot)\hat{f})^\vee(x)|E\|$ by

$$\sup \|(\varphi(\tau\cdot)\hat{f})^\vee(y)|E\|,$$

where sup is the supremum over

$$\{|x-y| \leq dt, t \leq \tau \leq 2t\}.$$

A remark on the proof (i)

Consists of two main parts:

1st step: Show that for two different φ_0 's and φ 's with the necessary conditions the two maximal function norms $\|\cdot\|_{B_{p,q}^s(E)}^*$ are equivalent.

A remark on the proof (i)

Consists of two main parts:

1st step: Show that for two different φ_0 's and φ 's with the necessary conditions the two maximal function norms $\|\cdot\|_{B_{p,q}^s(E)}^*$ are equivalent.

2nd step: Show that for fixed φ the maximal function norm $\|\cdot\|_{B_{p,q}^s(E)}^*$ can be estimated from above by $\|\cdot\|_{B_{p,q}^s(E)}^\varphi$. (the converse estimate is trivial:

$$\|(\varphi_j \hat{f})^\vee(x)|E\| \leq \sup_{y \in \mathbb{R}^n} \frac{\|(\varphi(2^{-j}\cdot) \hat{f})^\vee(x-y)|E\|}{(1+2^j|y|)^a}.$$

A remark on the proof (ii)

The most important difference to Triebel's proof is the given characterization of $B_{p,q}^s(E)$. But there is/was one problem in the **2nd step** of Rychkov's proof (of the scalar case): One has to estimate

$$\begin{aligned}d_j &= (\varphi_j^* f)_a(x) = \sup_{y \in \mathbb{R}^n} \frac{\|(\varphi(2^{-j} \cdot) \hat{f})^\vee(x - y)|E\|}{(1 + 2^j|y|)^a} \\ &= \sup_{y \in \mathbb{R}^n} \frac{\|(\check{\varphi}_j * f)(x - y)|E\|}{(1 + 2^j|y|)^a}\end{aligned}$$

from above by $c \cdot 2^{jN}$, where N is an arbitrary natural number and c is independent of j .

A remark on the proof (iii)

This can be done if we assume a priori $f \in B_{p,q}^s(E)$ or $F_{p,q}^s(E)$: By the lift property and the Sobolev embeddings there is a $\sigma \in \mathbb{N}$ such that $g := ((1 + |\xi|^2)^{-\sigma} \hat{f})^\vee \in L_\infty(E)$.

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$$\begin{aligned}
 (\varphi_j^* f)_a(x) &= \sup_{y \in \mathbb{R}^n} \frac{\left\| \left(\varphi(2^{-j}\cdot)(1 + |\xi|^2)^{+\sigma}(1 + |\xi|^2)^{-\sigma} \hat{f} \right)^\vee(y) \right\|_E}{(1 + 2^j|x - y|)^a} \\
 &= \sup_{y \in \mathbb{R}^n} \frac{\left\| \left((\varphi(2^{-j}\cdot)(1 + |\xi|^2)^\sigma)^\vee * g \right)(y) \right\|_E}{(1 + 2^j|x - y|)^a} \\
 &\leq \|g\|_{L_\infty(E)} \cdot \|(\varphi(2^{-j}\cdot)(1 + |\xi|^2)^\sigma)^\vee\|_{L_1} \\
 &\leq C \|g\|_{L_\infty(E)} \cdot \sum_{|\alpha| \leq 2\sigma} \|2^{j|\alpha|} D^\alpha \check{\varphi}(2^j\cdot)\|_{L_1} \\
 &\leq C \cdot 2^{2j\sigma} \|g\|_{L_\infty(E)} \cdot \sum_{|\alpha| \leq 2\sigma} \|D^\alpha \check{\varphi}\|_{L_1}.
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if $a \geq K$ and K is a constant depending on the norm of f as an element in $\mathcal{S}'(\mathbb{R}^n, E)$

$$\|\varphi\|_{K,L} := \sup_{y \in \mathbb{R}^n} (1 + |y|^2)^{\frac{K}{2}} \sum_{|\alpha| \leq L} \|D^\alpha \varphi(y)\|.$$

A remark on the proof (v)

So here is how the crucial part of the proof proceeds:

- 1 Let $f \in \mathcal{S}'(\mathbb{R}^n, E)$ be given with finite norm $\|f\|_{B_{p,q}^s(E)}^\varphi$ for an admissible φ .

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- ② By the 2nd step the maximal function norm $\|f\|_{B_{p,q}^s(E)}^*$ can be estimated from above by $\|f\|_{B_{p,q}^s(E)}^\varphi$ if $a > K$ (and K depends on f) and hence is finite as well.

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- ③ By the 1st step the function is in $B_{p,q}^s(E)$ (see the definition of $B_{p,q}^s(E)$).

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- 1 Let $f \in \mathcal{S}'(\mathbb{R}^n, E)$ be given with finite norm $\|f\|_{B_{p,q}^s(E)}^\varphi$ for an admissible φ .
- 2 By the **2nd step** the maximal function norm $\|f\|_{B_{p,q}^s(E)}^*$ can be estimated from above by $\|f\|_{B_{p,q}^s(E)}^\varphi$ if $a > K$ (and K depends on f) and hence is finite as well.
- 3 By the **1st step** the function is in $B_{p,q}^s(E)$ (see the definition of $B_{p,q}^s(E)$).
- 4 Then we have the necessary estimate of $d_j = (\varphi_j^* f)_a(x)$ **for all admissible a** and hence the **2nd step** can be applied once more for all $a > \frac{n}{p}$.

Vector-valued local means

Let $k_0, k^0 \in \mathcal{S}(\mathbb{R}^n)$, $\widehat{k}_0(0) \neq 0$, $\widehat{k}^0(0) \neq 0$, $N \in \mathbb{N}_0$ with $2N > s$ and $k^N := \Delta^N k^0$.

Then $\varphi_0 := \widehat{k}_0$ und $\varphi := \widehat{k^N}$ fulfil the conditions of the theorem.
In particular, k_0 and k^0 can be chosen with support in $\{|x| \leq 1\}$.

Vector-valued local means

Let $k_0, k^0 \in \mathcal{S}(\mathbb{R}^n)$, $\widehat{k}_0(0) \neq 0$, $\widehat{k}^0(0) \neq 0$, $N \in \mathbb{N}_0$ with $2N > s$ and $k^N := \Delta^N k^0$.

Then $\varphi_0 := \widehat{k}_0$ and $\varphi := \widehat{k}^N$ fulfil the conditions of the theorem. In particular, k_0 and k^0 can be chosen with support in $\{|x| \leq 1\}$. Let $k_j^N = 2^{jn} k^N(2^j \cdot)$.

Theorem (local means)

Let $2N > s$, $0 < p \leq \infty$ and $0 < q \leq \infty$. Then

$$\|f\|_{B_{p,q}^s(E)} \|k_0, k^N\| := \|k_0 * f\|_{L_p(E)} + \left(\sum_{j=1}^{\infty} 2^{jsq} \|k_j^N * f\|_{L_p(E)}^q \right)^{\frac{1}{q}}$$

is an equivalent norm for $B_{p,q}^s(E)$. It holds

$$B_{p,q}^s(E) = \{f \in \mathcal{S}'(\mathbb{R}^n, E) : \|f\|_{B_{p,q}^s(E)} \|k_0, k^N\| < \infty\}.$$

The end

Thank you for your attention

Questions?