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Generalized module of smoothness, K-functionals and approximation methods

I. Fourier means $(1 \le p \le \infty)$

• $\varphi : \mathbb{R}^d \longrightarrow \mathbb{C}$ is continous, • $\varphi(-\xi) = \overline{\varphi(\xi)}$, • $\varphi(\xi) = 0$ for $|\,\xi\,| \ge 1$, • $\varphi(0) = 1$ ($\varphi \in \mathcal{K}$ - *generator of method*)

$$W_0(\varphi)(h) \equiv 1\,,\ W_\sigma(\varphi)(h) = \sum_{k \in \mathbb{Z}^d} \varphi(k/\sigma)\, e^{ikh}\,,\ \sigma > 0\,, \quad (\textit{kernels})$$

$$\mathcal{F}_\sigma^{(\varphi)}(f;x) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(h)\, W_\sigma(\varphi)(x-h)\, dh\,, \ (\sigma \ge 0)$$

$$\boxed{\mathcal{F}_\sigma^{(\varphi)}(f;x) \;=\; \sum_{k \in \mathbb{Z}^d} \varphi(k/\sigma)\, f^\wedge(k)\, e^{ikx}\,,\ \sigma > 0}$$

Convergence in $\boldsymbol{L_p}$ $\forall\ \mathbf{1 \le p \le +\infty} \;\Leftrightarrow\; \mathbf{1} \in \boldsymbol{\mathcal{P}_\varphi} := \{\boldsymbol{p} \in (\mathbf{0}, +\infty] : \widehat{\varphi} \in \boldsymbol{L_p}(\mathbb{R}^d)\}$

Examples: Fejer, Vallee-Poussin, Rogosinski, Bochner-Riesz, Riesz, Blackman-Hamming, Favard.

II. K-functionals $(1 \le p \le \infty)$

• $\psi : \mathbb{R}^d \longrightarrow \mathbb{C}$ is continous and infinitely differntiable on $\mathbb{R}^d \setminus \{0\}$, • $\psi(-\xi) = \overline{\psi(\xi)}$, • $\psi(\tau\xi) = \tau^\alpha \psi(\xi)$, for $\tau > 0$, $\xi \in \mathbb{R}^d$, • $\psi(\xi) \neq 0$ for $\xi \neq 0$ *($\psi \in \mathcal{H}_\alpha$, $\alpha > 0$ - generator of smoothness)*

$$K_\psi(f, \delta)_p = \inf_{g \in X_p(\psi)} \{ \| f - g \|_p + \delta^\alpha \| \mathcal{D}(\psi) g \|_p \}, \; (\delta \ge 0)$$

$$\boxed{\mathcal{D}(\psi) g(x) = \sum_{k \in \mathbb{Z}^d} \psi(k) \, g^\wedge(k) \, e^{ikx}} \qquad (\textit{operator})$$

$X_p(\psi) = \{ g \in L_p : \mathcal{D}(\psi) g \in L_p \}$ *(ψ-smooth functions)*

Examples: usual derivatives, Weyl, Riesz, Laplace-operator Δ, its fractional powers $(-\Delta)^{\alpha/2}$.

Theorem 1 (quality of approximation). *Let* $1 \le p \le +\infty$, $\varphi \in \mathcal{K}$, $\varphi(\xi) \ne 1$ *for* $\xi \ne 0$, $1 \in \mathcal{P}_\varphi$ *and* $\psi \in \mathcal{H}_\alpha$ *for some* $\alpha > 0$. *If* $1-\varphi(\cdot) \overset{(1,\eta)}{\asymp} \psi(\cdot)$ *and* $(\varphi(\cdot))^m \overset{(1,\theta)}{\prec} 1-\varphi(\cdot)$ *for some* η, θ *and* $m \in \mathbb{N}$,

$$\| f - \mathcal{F}_\sigma^{(\varphi)}(f) \|_p \asymp K_\psi(f, 1/(\sigma+1))_p .$$

Notations: • $v(\cdot) \overset{(q,\eta)}{\prec} w(\cdot)$: $\mathcal{F}((\eta v)/w) \in L_q(\mathbb{R}^d)$, • $v(\cdot) \overset{(q,\eta)}{\asymp} w(\cdot)$, • (η, θ) : $\eta(\xi) = 1$ for $|\,\xi\,| \le \rho$, $\theta(\xi) = 1$ for $2\rho \le |\,\xi\,| \le 1$ and $\eta(\xi) + \theta(\xi) = 1$ for all $|\,\xi\,| \le 1$.

Means	***Operator***	***The result by***
Fejer	Riesz	*K. Ivanov, Z. Ditzian*
Rogosinski	second derivative	*R. Trigub*
Bochner-Riesz	Laplace-operator	*Z. Ditzian*
$\varphi_k(\xi) = (1 + i\xi^{2k+1})\eta(\xi)$	$(2k+1)$th-derivative, $k \in \mathbb{N}$	*K. Runovski, H.-J. Schmeisser*

III. Families of linear polynomial operators and polynomial K-functionals $(0 < p \le \infty)$

$$\mathcal{L}_{\sigma;\lambda}^{(\varphi)}(f;x) = (2N+1)^{-d} \cdot \sum_{\nu=0}^{2N} f\,(t_N^{\nu} + \lambda) \cdot W_{\sigma}(\varphi)\,(x - t_N^{\nu} - \lambda)$$

$$\lambda \in \mathbb{R}^d;\ N = [r\sigma],\ r \ge r(\varphi);\ t_N^{\nu} = \frac{2\pi\nu}{2N+1},\ \sum_{\nu=0}^{2N} \equiv \sum_{\nu_1=0}^{2N} \dots \sum_{\nu_d=0}^{2N}$$

Convergence in $\boldsymbol{L_p}$ *(in the sence* $\|\cdot\|_{\overline{p}} = (2\pi)^{-d/p} \| \ \|\cdot\|_{p;\,x} \|_{p;\,\lambda})$ $\Leftrightarrow$ $\boldsymbol{p \in \mathcal{P}_{\varphi}}$ ***(if*** $\boldsymbol{1 \in \mathcal{P}_{\varphi}}$***)***

$$\mathcal{K}_{\psi}(f,\delta)_p = \inf_{T \in \mathcal{T}_{1/\delta}} \{ \| f - T \|_p + \delta^{\alpha} \| \mathcal{D}(\psi) T \|_p \},\ (\delta > 0)$$

$$\mathcal{T}_{\sigma} = \left\{ T(x) = \sum_{k \in \mathbb{Z}^d} c_k\, e^{ikx} \ : c_{-k} = \overline{c_k},\ |\,k\,| \le \sigma \right\}$$

Theorem 1' (quality of approximation). *Let $0 < p \le +\infty$, $\varphi \in \mathcal{K}$, $\varphi(\xi) \ne 1$ for $\xi \ne 0$, $\widehat{\varphi} \in L_{\widetilde{p}}(\mathbb{R}^d)$ and $\psi \in \mathcal{H}_\alpha$ for some $\alpha > 0$. If $1 - \varphi(\cdot) \overset{(\widetilde{p},\eta)}{\asymp} \psi(\cdot)$ and $(\varphi(\cdot))^m \overset{(\widetilde{p},\theta)}{\prec} 1 - \varphi(\cdot)$ for some η, θ and $m \in \mathbb{N}$,*

$$\| f - \mathcal{L}_{\sigma;\lambda}^{(\varphi)}(f) \|_{\overline{p}} \asymp \mathcal{K}_\psi(f, 1/(\sigma+1))_p \,.$$

Notations: • $\widetilde{p} = \min(1,p)$.

IVa. Module of smoothness (special cases)

- $\omega_k(f,\delta)_p = \sup_{0\le h\le\delta} \left\| \sum_{\nu=1}^{k} (-1)^{k+\nu} \binom{k}{\nu} f(x+\nu h) - f(x) \right\|_p$
- $\omega_\alpha(f,\delta)_p$: $k \in \mathbb{N} \to \alpha > 0$ (*M. Potapov, S. Tikhonov*)
- $\widetilde{\omega}(f,\delta)_p$: $\ldots \frac{1}{4}(f(x\pm h, y) + f(x, y\pm h)) \ldots$ (*Z. Ditzian*)

Observations:

- modulus: smothness in terms of values of a function,
- $T_h - I$ is of multiplier type with a periodic generator,
- if θ is periodic, $\theta(0) = 0$, $\theta^\wedge(0) = -1$,

$$T_h f(x) - f(x) = \sum_{k\in\mathbb{Z}} \theta(kh)\, f^\wedge(k)\, e^{ikx} = \sum_{k\in\mathbb{Z}} \boldsymbol{\theta^\wedge(k)}\, f(x+kh)$$

IVb. Module of smoothness (general construction)

• $\theta : \mathbb{R} \longrightarrow \mathbb{C}$ is continuous and 2π-periodic, • $\theta(-\xi) = \overline{\theta(\xi)}$, • $\theta(\xi) \neq 0$ for $\xi \neq 2\pi m$, • $\theta(0) = 0$, • $\theta^\wedge(0) = -1$ ($\theta \in \Theta$ - *generator of modulus*)

$$\omega_\theta(f, \delta)_p = \sup_{0 \le h \le \delta} \left\| \sum_{k \in \mathbb{Z}} \theta^\wedge(k)\, f(x + kh) \right\|_p , \ (\, \delta \ge 0\,) .$$

It is well-defined (in the sense $\omega_\theta(f, \delta)_p \le C\| f \|_p$) for $p \in P_\theta := \left\{ p \in (0, +\infty] : \ \sum_{k \in \mathbb{Z}} |\, \theta^\wedge(k) \,|^p < +\infty \right\}$.

Theorem 2. ***Let*** $1 \le p \le +\infty$, $\theta \in \Theta$, $1 \in P_\theta$ ***and*** $\psi \in \mathcal{H}_\alpha$ ***for some*** $\alpha > 0$. ***If*** $\theta(\cdot) \overset{(1,\eta)}{\asymp} \psi(\cdot)$ ***for some*** η, ***then***

$$\omega_\theta(f, \delta)_p \asymp K_\psi(f, \delta)_p \,.$$

Example. (*K. Runovski, H.-J, Schmeisser*)

$$\widehat{\omega}(f,\delta)_p = \sup_{0\le h\le\delta} \left\| \frac{4}{\pi^2} \sum_{\nu\in\mathbb{Z}} \frac{f(x+(2\nu+1)h)}{(2\nu+1)^2} - f(x) \right\|_p, \ \delta \ge 0\,,$$

is generated by the 2π-periodic even function θ satisfying $\theta(\xi) = -(2/\pi)\,\xi$ for $\xi \in [0,\pi]$.

For $1 \le p \le +\infty$

$$\| \boldsymbol{f} - \mathcal{F}_{n-1}(\boldsymbol{f}) \|_p \asymp \boldsymbol{K}_{\langle ' \rangle}\left(\boldsymbol{f}, \boldsymbol{n}^{-1}\right)_p \asymp \widehat{\boldsymbol{\omega}}\left(\boldsymbol{f}, \boldsymbol{n}^{-1}\right)_p \,,$$

where

- F_n - *the Fejer means,*
- $K_{\langle ' \rangle}$ - *the K-functional related to the Riesz derivative.*

In particular (taking into account the Alexich-Zamanski theorem),

$$\boldsymbol{f} \in \widetilde{\mathbf{Lip}} \;\Leftrightarrow\; \widehat{\boldsymbol{\omega}}(\boldsymbol{f},\boldsymbol{\delta})_\infty = \boldsymbol{O}(\boldsymbol{\delta})\,.$$

How to describe the smoothness of an individual function?

- The smoothness is described by operators of multiplier type,
- the generator "embryo" $\zeta(\xi)$ is concentrated near 0 (close to a homogeneous function for the power scale),
- the describing terms depend on the extensions of "the embryo" to the real axis:

$$\begin{array}{ccc} \text{by } 1 & \longrightarrow & \text{polynomial approximation,} \\ \text{by homogeneity} & \longrightarrow & \text{differential operators } (K\text{-functionals}), \\ \text{by periodicity} & \longrightarrow & \text{discrete function values (module),} \end{array}$$

- all these descriptions are equivalent: ***if*** $\boldsymbol{\zeta(\cdot) \approx 1 - \varphi(\cdot) \approx \psi(\cdot) \approx \theta(\cdot)}$, $\boldsymbol{\varphi \in \mathcal{K}}$, $\boldsymbol{\psi \in \mathcal{H}_\alpha}$, $\boldsymbol{\theta \in \Theta}$, ***near*** $\mathbf{0}$, ***then*** ($\boldsymbol{\sigma \geq 1}$)

$$\| f - \mathcal{F}_{\sigma^{-1}}^{(\varphi)}(f) \|_p \asymp K_\psi(f, \sigma^{-1})_p \asymp \omega_\theta(f, \sigma^{-1})_p \,.$$

The further development

- the case $0 < p < 1$,
- the d-dimensional case,
- approximation by band-limited functions