Wavelet characterisation of Besov-Morrey and Triebel-Lizorkin-Morrey spaces

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Definition

For $0 < u \leq p \leq \infty$ the spaces

$$M_{p,u} := M_{p,u}(\mathbb{R}^n) := \{f \in L_u^{loc}(\mathbb{R}^n) : \|f|M_{p,u}\| < \infty\}$$

are called Morrey spaces, their quasi-norms $\|\cdot|M_{p,u}\|$ for $u < \infty$ are defined as

$$\begin{aligned} \|f|M_{p,u}\| &:= \sup_{R>0, x\in\mathbb{R}^n} R^{n\left(\frac{1}{p}-\frac{1}{u}\right)} \left(\int_{B_R(x)} |f(y)|^u \, dy\right)^{\frac{1}{u}} \\ &= \sup_{R>0, x\in\mathbb{R}^n} R^{n\left(\frac{1}{p}-\frac{1}{u}\right)} \|f\|_{L_u(B_R(x))}. \end{aligned}$$

For $u = \infty$ we define the norm $\|\cdot|M_{\infty,\infty}\|$ analogue by using the essential supremum.

- If $p > u$ then $M_{p,u} = \{0\}$.
- $\|f|M_{p,p}\| = \|f\|_{L_p}$

Theorem

1. $(M_{p,u}, \|\cdot|M_{p,u}\|)$ *is a quasi-Banach space and if* $u \geq 1$ *a Banach space.*
2. *If* $p = \infty$ *then* $M_{\infty,u} = L_\infty$ *with norm equivalence.*
3. *If* $0 < u' \leq u \leq p$ *then* $M_{p,u} \hookrightarrow M_{p,u'}$. *In particular* $L_p \hookrightarrow M_{p,u'}$ *for* $0 < u' \leq p$.
4. *If* $0 < u < p$ *then* $w - L_p \hookrightarrow M_{p,u}$ *where this inclusion is strict.* *(Recall:* $\|f|w - L_p\| := \sup_{t>0} (t^p \lambda_f(t))^{\frac{1}{p}}$ *where* $\lambda_f(t) := |\{x : |f(x)| > t\}|$*)*
5. *If* $f \in L_\infty \cap L_u$ *then* $f \in M_{p,u}$.
6. *If* $0 < u < p$ *then* $L\infty \cap M_{p,u}$ *is not dense in* $M_{p,u}$. *(Particularly the Schwartz space* S *is not dense in* $M_{p,u}$.*)*

1.2. Elementar properties and inclusions

Theorem (Piccini 1969, thesis)

$0 < u \leq p < \infty$, $0 < u' \leq p' < \infty$. *If* $M_{p,u} \subset M_{p',u'}$ *then* $0 < u' \leq u \leq p' = p$.

Remark

1. *If* $L_\infty \subset M_{p,u}$ *then* $p = \infty$.
2. *If* $M_{p,u} \subset L_\infty$ *then* $p = \infty$.

Corollary

$M_{p,u} \hookrightarrow M_{p',u'}$ *if and only if either* $0 < u' \leq u \leq p' = p < \infty$ *or* $0 < u', u \leq p' = p = \infty$.

Definition

$f \in L_1^{loc}(\mathbb{R}^n)$. The Hardy-Littlewood maximal function Mf is defined as

$$Mf(y) := \sup_{R>0} \frac{1}{\mu(B_R(y))} \int_{B_R(y)} |f(z)|\, dz.$$

$\mathcal{L}(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{C} \text{ measurable}\}$, $0 < u \leq p \leq \infty$. If $0 < q < \infty$ then

$$M_{p,u}(\ell_q) := \{(f_j)_j \subset \mathcal{L}(\mathbb{R}^n) : \left(\sum_{j=0}^{\infty} |f_j|^q\right)^{\frac{1}{q}} \in L_u^{loc}(\mathbb{R}^n) \text{ and } \|f_j|M_{p,u}(\ell_q)\| < \infty\}$$

is called vector-valued Morrey space, where their norm is defined by

$$\|f_j|M_{p,u}(\ell_q)\| := \| \left(\sum_{j=0}^{\infty} |f_j(\cdot)|^q\right)^{\frac{1}{q}} |M_{p,u}\|.$$

For $q = \infty$ natural modification.

Theorem (vector-valued maximal inequality, Tang/Xu 2005)

Let $1 < u \leq p < \infty$, $1 < q \leq \infty$. If $(f_j)_{j=0}^{\infty}$ is a sequence measurable functions then

$$\|Mf_j|M_{p,u}(\ell_q)\| \leq c\,\|f_j|M_{p,u}(\ell_q)\|\,.$$

Remark

The proof relies on the vector-valued Fefferman-Stein maximal inequality (1971) and a decomposition of f_j in annuli in the manner $f_j^0 := \chi_{B_{2R}(x)} f_j$ and $f_j^i := \chi_{B_{2^{i+1}R}(x)\setminus B_{2^iR}(x)} f_j$ for $i \geq 1$. The scalar-valued maximal inequality is due to Chiarenza/Frasca 1987.

2.2 Fourier multiplier theorem

Definition

Denote by F the Fourier transform in $S'(\mathbb{R}^n)$. For $\Omega \subset \mathbb{R}^n$ compact and $0 < u \leq p \leq \infty$ we set

$$M^{\Omega}_{p,u}(\mathbb{R}^n) := \{f \in M_{p,u}(\mathbb{R}^n) \cap S'(\mathbb{R}^n) : \operatorname{supp} Ff \subset \Omega\} .$$

Lemma

$\Omega \subset \mathbb{R}^n$ compact, diameter d from Ω and $0 < u \leq p \leq \infty$. Then exists a from Ω independent constant c, so that

$$\|f|L_\infty\| \leq c\, d^{\frac{n}{u}}\, \|f|M_{p,u}\|$$

for all $f \in M^{\Omega}_{p,u}$.

2.2 Fourier multiplier theorem

Definition

$\nu \in \mathbb{R}$. The fractional Sobolev space is defined as

$$H_2^\nu(\mathbb{R}^n) := \left\{ f \in S'(\mathbb{R}^n) : \|f|H_2^\nu\| := \left\| \left(1 + |\cdot|^2\right)^{\frac{\nu}{2}} Ff(\cdot)|L_2 \right\| \right\}.$$

Definition

We define for $M \in H_2^\nu(\mathbb{R}^n)$ so that $\nu > \frac{n}{2}$ and $f \in M_{p,u}^\Omega$

$$F^{-1}[MFf](x) := (2\pi)^{-n} \int_{\mathbb{R}^n} (F^{-1}M)(x-y) f(y) dy.$$

2.2 Fourier multiplier theorem

Theorem (Tang/Xu 2005, Sawano/Tanaka 2007)

$0 < u \leq p \leq \infty$, $0 < q \leq \infty$, $(\Omega_j)_{j \in \mathbb{N}_0}$ *a sequence compact sets,* $f_j \in M^{\Omega_j}_{p,u}$, $M_j \in H^\nu_2(\mathbb{R}^n)$, *the diameter* $d_j > 0$ *from* Ω_j *for* $j \in \mathbb{N}_0$ *and* $\nu > \frac{n}{2} + \frac{n}{\min\{u,q\}}$.

1. *If additionally* $p < \infty$ *then exists a constant* c *so that*

$$\left\| F^{-1}[M_j F f_j] | M_{p,u}(\ell_q) \right\| \leq c \sup_j \| M_j(d_j \cdot) | H^\nu_2 \| \, \| f_j | M_{p,u}(\ell_q) \|$$

2. *It exists a constant* c *so that*

$$\left(\sum_{j=0}^{\infty} \left\| F^{-1}[M_j F f_j] | M_{p,u} \right\|^q \right)^{\frac{1}{q}} \leq c \sup_j \| M_j(d_j \cdot) | H^\nu_2 \| \left(\sum_{j=0}^{\infty} \| f_j | M_{p,u} \|^q \right)^{\frac{1}{q}}$$

The constants c *are independent from* $(M_j)_j$, $(f_j)_j$ *and* $(d_j)_j$.

Definition

We denote a sequence of functions $(\varphi_j)_{j\in\mathbb{N}_0} \subset C_0^\infty(\mathbb{R}^n)$ as smooth dyadic resolution of the unit if:

$$\text{supp } \varphi_0 \subset \{\xi : |\xi| \leq 2\},$$
$$\text{supp } \varphi_j \subset \left\{\xi : 2^{j-1} \leq |\xi| \leq 2^{j+1}\right\} \text{ for all } j \in \mathbb{N},$$

for all multi-index $\alpha \in \mathbb{N}_0^n$ exist a constant c_α so that

$$|D^\alpha \varphi_j(\xi)| \leq c_\alpha 2^{-j|\alpha|} \text{ for all } j \in \mathbb{N} \text{ and all } \xi \in \mathbb{R}^n$$

and

$$\sum_{j=0}^{\infty} \varphi_j(\xi) = 1 \text{ for all } \xi \in \mathbb{R}^n.$$

Definition

$0 < q < \infty$, $s \in \mathbb{R}$ and $(\varphi_j)_j$ a smooth dyadic resolution of the unit. If $0 < u \leq p \leq \infty$ then the spaces

$$MB^s_{(pu),q} := \left\{ f \in S'(\mathbb{R}^n) : \left\| f | MB^s_{(pu),q} \right\|_\varphi < \infty \right\}$$

are called Besov-Morrey spaces, where their quasi-norms are defined as

$$\left\| f | MB^s_{(pu),q} \right\|_\varphi := \left\| 2^{js} F^{-1} \left[\varphi_j F f \right] (\cdot) | \ell_q(M_{pu}) \right\| .$$

If $0 < u \leq p < \infty$ then the spaces

$$MF^s_{(pu),q} := \left\{ f \in S'(\mathbb{R}^n) : \left\| f | MF^s_{(pu),q} \right\|_\varphi < \infty \right\}$$

are called Triebel-Lizorkin-Morrey spaces, where their quasi-norms are defined as

$$\left\| f | MF^s_{(pu),q} \right\|_\varphi := \left\| 2^{js} F^{-1} \left[\varphi_j F f \right] (\cdot) | M_{pu}(\ell_q) \right\| .$$

Definition

1. Let $\nu \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$. Then we define dyadic cubes

$$Q_{\nu m} := \prod_{j=1}^{n} \left[\frac{m_j}{2^\nu}, \frac{m_j + 1}{2^\nu} \right) = 2^{-\nu} m + 2^{-\nu} [0, 1)^n$$

and for $0 < p \leq \infty$ the p-normalized indicator

$$\chi_{\nu,m}^{(p)} := 2^{\frac{n\nu}{p}} \chi_{Q_{\nu m}}.$$

2. Let $0 < u \leq p \leq \infty$, $0 < q \leq \infty$. Then, given a doubly indexed complex sequence $\lambda = (\lambda_{\nu m})_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$, set

$$\left\| \lambda | n_{(pu),q}^{s} \right\| := \left(\sum_{\nu \in \mathbb{N}_0} 2^{\nu (s - \frac{n}{p}) q} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} \chi_{\nu,m}^{(p)}(\cdot) \right| M_{pu} \right\|^q \right)^{\frac{1}{q}}.$$

Definition

Let $K \in \mathbb{N}_0$, $L \in \mathbb{N}_0$ and $d \geq 1$. Then the L_∞-functions $a_{\nu m} : \mathbb{R}^n \mapsto \mathbb{C}$ are called $[K, L, d]$-atoms centered at $Q_{\nu m}$, $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, if

$$supp\ a_{\nu m} \subset dQ_{\nu m}, \ \nu \in \mathbb{N}_0, \ m \in \mathbb{Z}^n;$$

there exists all (classical) derivatives $D^\alpha a_{\nu m}$ with $|\alpha| \leq K$ such that

$$|D^\alpha a_{\nu m}(x)| \leq 2^{\nu|\alpha|}, \ |\alpha| \leq K, \ \nu \in \mathbb{N}_0, \ m \in \mathbb{Z}^n \tag{1}$$

for all $x \in \mathbb{R}^n$ and

$$\int_{\mathbb{R}^n} x^\beta a_{\nu m}(x) dx = 0, \ |\beta| < L, \ \nu \in \mathbb{N}, \ m \in \mathbb{Z}^n. \tag{2}$$

For $\nu = 0$ no moment condition (4) is required. Furthermore, if $L = 0$ then (4) is empty (/ no condition). If $K = 0$ then (3) means $a_{\nu m} \in L_\infty$ and $|a_{\nu m}(x)| \leq 1$.

Atomic decomposition [Sawano/Tanaka 2007]

Let $0 < u \leq p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$. Let $K \in \mathbb{N}_0$, $L \in \mathbb{N}_0$ with

$$K > s,\ L > \sigma_u - s, \sigma_u := n\left(\frac{1}{\min(1,u)} - 1\right).$$

Further let $d \in \mathbb{R}$ with $d \geq 1$. If $(a_{\nu m})_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ are $[K, L, d]$-atoms centered at $Q_{\nu m}$ and $\lambda = (\lambda_{\nu m})_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in n^s_{(pu),q}$, then

$$\sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}$$

converges unconditional in S'. Moreover, the series converges unconditional in $MB^{\sigma}_{(pu),q}(K)$ for any ball K in $\mathbb{R}^n$ for all $\sigma < s$. (If in addition $p = u$ and $p, q < \infty$, then the convergence is even in $MB^s_{(pp),q}$ (also unconditional).) Moreover,

$$\left\| \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m} \middle| MB^s_{(pu),q} \right\| \leq c \left\| \lambda \middle| n^s_{(pu),q} \right\|.$$

Here c is a constant independent of $(\lambda_{\nu m})_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ and $(a_{\nu m})_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$.

Atomic decomposition [Sawano/Tanaka 2007]

Let $0 < u \leq p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$. Let $K \in \mathbb{N}_0$, $L \in \mathbb{N}_0$ with

$$L > \sigma_u - s, \sigma_u := n \left(\frac{1}{\min(1,u)} - 1 \right),$$

then for each $f \in MB^s_{(pu),q}$ exists $\lambda = (\lambda_{\nu m})_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in n^s_{(pu),q}$ and $[K, L, d]$-atoms $(a_{\nu m})_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ centered at $Q_{\nu m}$ (with $d = 2\sqrt{n}$) such that the representation

$$f = \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}$$

holds, whereas the equality and the convergence of the series is in $S'(\mathbb{R}^n)$. The convergence is even unconditional in S'. Moreover, the series converges unconditional in $MB^\sigma_{(pu),q}(K)$ for any ball K in $\mathbb{R}^n$ for all $\sigma < s$. (For $p = u$ and $p, q < \infty$ the convergence is even in $MB^s_{(pp),q}$ (also unconditional).) Furthermore,

$$\left\| \lambda | n^s_{(pu),q} \right\| \leq c \left\| f \,|\, MB^s_{(pu),q} \right\|,$$

where the constant c is universal for all $f \in MB^s_{(pu),q}$.

Definition

Let $A \in \mathbb{N}_0$, $B \in \mathbb{N}_0$ *and* $C > 0$. *Then the* L_∞*-functions* $k_{\nu m} : \mathbb{R}^n \mapsto \mathbb{C}$ *with* $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, *are called kernels (of local means) centered at* $Q_{\nu m}$ *if*

$$\mathrm{supp}\ k_{\nu m} \subset CQ_{\nu m},\ \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n;$$

there exists all (classical) derivatives $D^\alpha k_{\nu m}$ *with* $|\alpha| \leq A$ *such that*

$$|D^\alpha k_{\nu m}(x)| \leq 2^{\nu n + \nu|\alpha|},\ |\alpha| \leq A,\ \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n \tag{3}$$

for all $x \in \mathbb{R}^n$ *and*

$$\int_{\mathbb{R}^n} x^\beta k_{\nu m}(x) dx = 0,\ |\beta| < B,\ \nu \in \mathbb{N},\ m \in \mathbb{Z}^n. \tag{4}$$

For $\nu = 0$ *no moment condition (4) is required. Furthermore, if* $B = 0$ *then (4) is empty (/ no condition). If* $A = 0$ *then (3) means* $k_{\nu m} \in L_\infty$ *and* $|k_{\nu m}(x)| \leq 2^{\nu n}$.

Definition

*Let $f \in MB^{s}_{(pu),q}$ where $0 < u \le p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. Let $k_{\nu m}$ be kernels with $A > \sigma_u - s$, $B \in \mathbb{N}_0$ and $k_{\nu m} \in C^k$ **with** $k > \frac{n}{u} - s$. Then*

$$f(k_{\nu m}) := \langle k_{\nu m}, f\rangle := \int_{\mathbb{R}^n} k_{\nu m}(y) f(y) dy, \ \nu \in \mathbb{N}, \ m \in \mathbb{Z}^n, \qquad (5)$$

are local means, considered as a dual pairing within $\langle S, S'\rangle$. Furthermore,

$$f(k) := \{ f(k_{\nu m}) | \, \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n \}.$$

$MB^{s}_{(pu),q} \hookrightarrow B^{s-\frac{n}{u}}_{\infty q} \hookrightarrow \left(B^{-s+\frac{n}{u}}_{1,1}\right)'$ whereas we have that S is dense in $B^{-s+\frac{n}{u}}_{1,1}$. Therefor a linear functional on $B^{-s+\frac{n}{u}}_{11}$ can be interpreted as an element of S'. We set $f \in \left(B^{-s+\frac{n}{u}}_{1,1}\right)'$, if and only if there exists a constant c such that $|f(\varphi)| \le c \left\| \varphi | \, B^{-s+\frac{n}{u}}_{1,1} \right\|$ for all $\phi \in S$. Moreover, $k_{\nu m} \in B^{-s+\frac{n}{u}}_{1,1}$ holds by $k_{\nu m} \in C^k$ with $k > \frac{n}{u} - s$ and the compact support of $k_{\nu m}$.

Theorem

*Let $0 < u \leq p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$. Let $k_{\nu m}$ be kernels with $k_{\nu m} \in C^k$ **with** $k > \frac{n}{u} - s$ and $A \in \mathbb{N}_0$, $B \in \mathbb{N}_0$ with*

$$A > \sigma_u - s, B > s, \sigma_u := n\left(\frac{1}{min(1,u)} - 1\right),$$

and $C > 0$ are fixed. Then

$$\left\| f(k) | n^s_{(pu),q} \right\| \leq c \left\| f | MB^s_{(pu),q} \right\|,$$

where the constant c is universal for all $f \in MB^s_{(pu),q}$.

Proof.

The ideas of the proof follows Theorem 1.15 of the book „Function Spaces and Wavelets on Domains“ from Triebel 2008. □

3.4 Wavelets

Recall that a wavelet is a function $\psi(x) \in L_2(\mathbb{R})$ such that the family of functions $\left\{2^{\frac{\nu}{2}}\psi(2^\nu x - m)\right\}_{\nu\in\mathbb{Z},m\in\mathbb{Z}}$ is an orthonormal basis in $L_2(\mathbb{R})$. If you win the wavelet with the help of a multiresolution analysis you get the orthonormal basis also with $\left\{\psi_F(x-m), 2^{\frac{\nu}{2}}\psi_M(2^\nu x - m)\right\}_{\nu\in\mathbb{N},m\in\mathbb{Z}}$ where ψ_F is called scaling funtion (or father wavelet) and ψ_M the associated wavelet (or mother wavelet).
For any $u \in \mathbb{N}$ there exists a real compactly supported scaling function $\psi_F \in C^u$ and a real compactly supported associated wavelet $\psi_M \in C^u$ such that $(F\psi_F)(0) = (2\pi)^{-\frac{1}{2}}$ and

$$\int_{\mathbb{R}} x^v \psi_M(x)dx = 0 \text{ for all } v \in \mathbb{N}_0 \text{ with } v < u.$$

These wavelets are called Daubechies wavelets. We extend these wavelets from $\mathbb{R}$ to $\mathbb{R}^n$ by the usual tensor procedure. Let

$$G := (G_1, \dots, G_n) \in G^0 := \{F, M\}^n ,$$

which means that G_r is either F or M. Let

$$G := (G_1, \dots, G_n) \in G^\nu := \{F, M\}^{n*} , \ \nu \in \mathbb{N},$$

which means that G_r is either F or M where $*$ indicates that at least one of the components of G must be an M. Hence G^0 has 2^n elements, whereas G^ν with $\nu \in \mathbb{N}$ has $2^n - 1$ elements. Let

$$\psi^\nu_{G,m} := 2^{\nu \frac{n}{2}} \prod_{r=1}^{n} \psi_{G_r}(2^\nu x_r - m_r), \; G \in G^\nu, \; m \in \mathbb{Z}^n,$$

where $\nu \in \mathbb{N}_0$. We always assume that ψ_F, ψ_M have L_2-norm 1. Then

$$\{ \psi^\nu_{Gm} | \, \nu \in \mathbb{N}_0, \; G \in G^\nu, \; m \in \mathbb{Z}^n \}$$

is an orthonormal basis in $L_2(\mathbb{R}^n)$ for any $u \in \mathbb{N}$ and

$$f = \sum_{\nu \in \mathbb{N}_0} \sum_{G \in G^\nu} \sum_{m \in \mathbb{Z}^n} \langle f, \psi^\nu_{Gm} \rangle \, \psi^\nu_{Gm}$$

in L_2. We recall for a complex quasi-Banach space that $\{b_j\}_j \subset B$ is called a (Schauder) basis if and $b \in B$ can be uniquely represented as $b = \sum_{j \in \mathbb{N}} \lambda_j b_j$, $\lambda_j \in \mathbb{C}$ with convergence in B. The basis is called unconditional if for any rearrangement σ, $\left\{ b_{\sigma(j)} \right\}_j$ is again a basis with the same outcome.

Let $\nu \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$. Then we define dyadic cubes

$$Q_{\nu m} := \prod_{j=1}^{n} \left[\frac{m_j}{2^\nu}, \frac{m_j + 1}{2^\nu} \right) = 2^{-\nu} m + 2^{-\nu} [0, 1)^n$$

and for $0 < p \leq \infty$ the p-normalized indicator

$$\chi^{(p)}_{\nu,m} := 2^{\frac{n\nu}{p}} \chi_{Q_{\nu m}}.$$

Definition

Let $0 < u \leq p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$. Then $\bar{n}^s_{(pu),q}$ is the collection of all sequences

$$\lambda = \{ \lambda^\nu_{Gm} \in \mathbb{C} | \, \nu \in \mathbb{N}_0, \; G \in G^\nu, \; m \in \mathbb{Z}^n \}$$

such that

$$\left\| \lambda | \bar{n}^s_{(pu),q} \right\| := \left(\sum_{\nu \in \mathbb{N}_0} 2^{\nu \left(s - \frac{n}{p}\right) q} \sum_{G \in G^\nu} \left\| \left| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} \chi^{(p)}_{\nu,m}(\cdot) \right| M_{pu} \right\|^q \right)^{\frac{1}{q}}.$$

Theorem

Let $0 < u \leq p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ *and* ψ^{ν}_{Gm} *be the Daubechies wavelets with*

$$u > \max(s, n\left(\frac{1}{min(1,u)} - 1\right) - s, \frac{n}{u} - s).$$

Let $f \in S'$. *Then* $f \in MB^{s}_{(pu),q}$ *if, and only if, it can be represented as*

$$f = \sum_{\nu,G,m} \lambda^{\nu}_{Gm} 2^{-\nu\frac{n}{2}} \psi^{\nu}_{Gm}, \;\; \lambda \in \bar{n}^{s}_{(pu),q},$$

unconditional convergence being in S' *and locally in any space* $MB^{\sigma}_{(pu),q}$ *with* $\sigma < s$ *whereas the convergence follows by the representation and is not a addition condition for* $f \in MB^{s}_{(pu),q}$*).*

Theorem

The representation is unique,

$$\lambda^{\nu}_{Gm} = 2^{\nu \frac{n}{2}} \langle f, \psi^{\nu}_{Gm} \rangle$$

and

$$I : f \mapsto \left\{ 2^{\nu \frac{n}{2}} \langle f, \psi^{\nu}_{Gm} \rangle \right\}$$

is an isomorphic map of $MB^{s}_{(pu),q}$ *onto* $\bar{n}^{s}_{(pu),q}$*. But* $\{\psi^{\nu}_{Gm}\}$ *is not a basis in* $MB^{\sigma}_{(pu),q}$ *for all* $0 < u < p \leq \infty$ *and* $0 < u = p = \infty$ *(, therefore recall that* $\{\psi^{\nu}_{Gm}\}$ *is a basis in* $MB^{\sigma}_{(pp),q}$ *for all* $\max(p, q) < \infty$*.*

Proof.

The ideas of the proof follows Theorem 1.15 of the book „Function Spaces and Wavelets on Domains“ from Triebel 2008. □

3.5 Besov-Triebel-Lizorkin-Morrey spaces

- Littlewood-Paley-Theorem [Mazzucato 2003]: $MF^0_{(pu),2} = M_{pu}$ with norm-equivalence
- Various decompositions including wavelets, quarks, atoms and molecules [Sawano/Tanaka 2008]
- Characterisations by wavelets, differences, oscillations for the F-skale [Sickel/Yang/Yuan]

- Goal: Characterisations by wavelets for Besov-Triebel-Lizorkin-Morrey spaces on domains