Compact-defined decompositions of spaces with applications to calculus of variations in Sobolev spaces and Bochner integral in Frechet spaces.

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Last decade, at first the variational problems in Sobolev spaces, later the Radon-Nikodym problem in vector integration, and some other problems leads us to the new function characteristics, both scalar and convex compact, which are closely connected to a certain decomposition of the space under consideration.

That decomposition is generated by the Banach subspaces spanned by the all absolutely convex compacta from the initial space. The structure of such decompositions of Frechet spaces is like to one of the «diffracting screen» in physics. The spectrum of such subspaces doesn't change the linear effects, but shows evidently enough some new nonlinear effects.

It seems that the researches with the help of compact-defined decompositions can be used for many actual problems of the modern analysis. At least, my own program is extensive enough (may be, too much).

Some works about reference. The papers 1 — 5 are devoted mainly to variational problems. That researches were realized jointly with E.Bozhonok. The papers 6 — 9 are devoted mainly to vector integration. That researches were realized jointly with F.Stonyakin. Finally, the paper 10 contains compact analog of Banach-Zaretsky theorem.

Contents

1. Compact-defined decompositions of LCS

1.1. Def. Let E be a real complete LCS. Denote further by $\mathcal{C}(E) = \{C\}$ the system of all the absolutely convex compact sets in E; E_C are the linear spans of C, equipped with the norms $\|\cdot\|_C$ generated by C. Note that $(E_C, \ \|\cdot\|_C)$ are Banach spaces.
1.2. Rem. The system $\overrightarrow{E}_{\mathcal{C}} := \{E_C\}_{C\in\mathcal{C}(E)}$ forms an inductive spectrum of the Banach spaces respective to the continuous embeddings that corresponds to the order $C_1 \subset M \cdot C_2$. In addition, compactness of $C \in \mathcal{C}(E)$ implies continuity of the embeddings $(E_C, \|\cdot\|_C) \hookrightarrow E$. It follows, in turn, continuity of the embedding $E_{\mathcal{C}} = \varinjlim_{C} E_C \hookrightarrow E$ and coincidence of the supports $\mid E_{\mathcal{C}} \mid = \mid E \mid$. The inverse continuous embedding, in general, isn't true.
1.3. Ex. Let E be an arbitrary Banach space, E^* be it's dual, B_1 be the unit ball in E^*. According to Banach-Alaoglu theorem, B_1 is weakly-$*$-compact; moreover, B_1 absorbs the all other weak compactae in E^*. Let's pass to the weak-$*$ topology $\sigma(E^*, E)$ in E^*. Then $E^*_{B_1}$ absorbs the all other subspaces E^*_C, $C \in \mathcal{C}(E^*_\sigma)$, whence the equality $E^*_{\mathcal{C}} = E^*_{B_1}$ follows. So, the topology of $E^*_{\mathcal{C}}$ coincides with the initial strong-$*$ topology in E^*. That means discontinuity of the inverse embedding $E^*_\sigma \hookrightarrow (E^*_\sigma)_{\mathcal{C}}$
(for infinite-dimensional E).

However, the following fundamental result used as a base of our further research holds.
1.4. Theor. For an arbitrary Frechet space E the following equality is true:

$$E_{\mathcal{C}} := \varinjlim_{C\in\mathcal{C}(E)} E_C \overset{top}{=} E.$$

In fact, the last representation possesses considerably more strong properties. First, let's introduce a compact approximation property.
1.5. Def. Say that a locally convex space E satisfies *a compact approximation property* $E \in K_{ap}$ if for every $C \in \mathcal{C}(E)$ such $C' \in \mathcal{C}(E)$ exists that the compact embedding $E_C \hookrightarrow\hookrightarrow E_{C'}$ holds. Next, E satisfies a *σ-compact approximation property* $(E \in K^\sigma_{ap})$ if for every sequence $\{C_n\} \subset \mathcal{C}(E)$ such $C' \in \mathcal{C}(E)$ exists that all the embeddings $E_{C_n} \hookrightarrow\hookrightarrow$
$\hookrightarrow\hookrightarrow E_{C'}$ $(n = 1, 2, ...)$ are compact.

1.6. Theor. Every Frechet space satisfies σ-compact approximation property K^σ_{ap}). In other words, the system $\{E_C\}_{C\in\mathcal{C}(E)}$ forms *σ-inductive spectrum* of the Banach spaces respective to the compact embeddings.

Note in addition that such continuous map $\varphi : E \to E$ can be constructed that the all embeddings from E_C into $E_{\overline{co}\ \varphi(C)}$ are compact. So, the compact embeddings are built constructively.

2. Universal compacta. Compact ellipsoids.

The system of all absolutely convex compacta $\mathcal{C}(E)$ is, in general, very large. It is more convenient to minimize that system with a view to obtain a more simple compact decomposition of E.

2.1. Def. Call a subset $\mathcal{C}_u(E) \subset \mathcal{C}(E)$ the *universal compact system* in E, if each compact set from $\mathcal{C}(E)$ is absorbed by some compactum C' from $\mathcal{C}_u(E)$.

2.2. Rem. Let's return to the example 1.3: E is Banach space, E^*_σ is its dual space with weak-$*$ topology. Then the unit ball B_1 from E^* itself forms a universal compact system in E^*_σ (by Banach-Alaoglu theorem).

In general case, the situation is more complicated. First, consider the important case of infinite-dimensional separable Hilbert space $E = H$. In that case, the universal compact system in H is formed by *the compact ellipsoids*. Let's bring a definition.

2.3. Def. Let's fix some orthonormal basis $\{e_n\}_1^\infty$ in H and some sequence of the positive integers $\varepsilon = (\varepsilon_k > 0)_1^\infty$. *An ellipsoid* C_ε (respective to the basis) is defined as the following set

$$C_\varepsilon = \left\{ x = \sum_{k=1}^{\infty} x_k e_k \mid \sum_{k=1}^{\infty} \left(\mid x_k \mid^2 / \varepsilon_k^2 \right) \leqslant 1 \right\}. \qquad (*)$$

Obviously, all C_ε are closed and absolutely convex. The following criterion is well known.

2.4.Theor. C_ε is compact if and only if ε_k tends to 0.

2.5.Theor. The compact ellipsoids form *a universal compact system* in H.

Let's explain an analytical sense of K-ellipsoids. The condition above defines a velocity of tending to zero of the Fourier coefficients of x. That, in turn, as it's well known, defines a smoothness class of x. Let's bring corresponding examples.

2.6. Ex. (i) In case of $H = L_2[0; 2\pi]$, the basis $(e_k = e^{ikt}/\sqrt{2\pi})$ and

$\varepsilon_k = O\left(\frac{1}{|k|^m}\right)$ we get $x^{(m-1)} \in BV[0;2\pi]$.

(ii) $H = \overset{\circ}{H^1}[0;2\pi]$, the basis $(e_k = e^{ikt}/\sqrt{2\pi(k^2+1)})$ and the same ε_k we get $x^{(m)} \in BV[0;2\pi]$.

2.7. Rem. The compact ellipsoids as universal compacta can be introduced in some other spaces, such as the following.

(i) $\ell_p (1 \leqslant p < \infty)$: $C_\varepsilon = \left\{ x \mid \sum\limits_{k=1}^{\infty} | x_k |^p / \varepsilon_k^p \right\} \qquad (\varepsilon_k \to 0)$,

(ii) c_0: $C_\varepsilon = \left\{ x \mid \sup\limits_k (| x_k | / \varepsilon_k) \leqslant 1 \right\} \qquad (\varepsilon_k \to 0)$,

(iii) $C[a;b]$: $C_\varepsilon = \left\{ x \mid \max\left(| x(a) |, \sup\limits_k \frac{\omega_x(\delta_k)}{\varepsilon_k} \right) \leqslant 1 \right\} \quad (\varepsilon_k \to 0,\ \delta_k \text{ fixed})$.

3. Subspaces generated by compact ellipsoids.

3.1. Rem. First, note that for all compact ellipsoids C_ε the spaces H_{C_ε} are Hilbert weighted spaces with the norms

$$\|x\|^2_{C_\varepsilon} = \sum_{k=1}^{\infty} \left(| x_k |^2 / \varepsilon_k^2 \right).$$

Next, it follows from Th.2.5 immediately that

3.2.Th. The system $\overrightarrow{H}_{C_\varepsilon} = \{H_{C_\varepsilon} \mid \varepsilon_k \to 0\}$ forms an inductive spectrum of Hilbert spaces, whose inductive limit coincides with initial space.

Note also some other preferences of the passage to the spaces H_{C_ε}.

3.3. Th. The system $\overrightarrow{H}_{C_\varepsilon}$ forms an inductive spectrum respective to the embeddings of the class S_p for all $1 \leqslant p < \infty$. Moreover, H_{C_ε} are densely embedded into H.

3.4. Rem. Note, at last, that, by virtue of a known «inductive limit theorem», an arbitrary linear operator $A : H \to F$ is continuous iff all the restrictions of A onto H_{C_ε} are continuous. But it isn't the case for nonlinear operators, as it'll be seen further. The K-decomposition, in fact, represents a «trap» for nonlinear effects, not changing linear ones.

That phenomenon will be serve further as a base for applications to variational problems in H^1.

4. Calculus of variations in H^1: K-extremae and others.

Here we use K-decomposition for domain of the variational functionals of the type

$$\Phi(y) = \int_a^b f(x, y, y')dx \quad \left(y(\cdot) \in H^1[a; b]\right). \qquad (*)$$

First, let's introduce K-analytical characteristics of functionals connected to K-decomposition of the domain.

<u>4.1. Def.</u> Let E be an arbitrary LCS, $E = \varinjlim_{C \in \mathcal{C}(E)} E_C$ be its K-representation, Φ be a real functional on E. The following K-properties are introduced as usual ones for the all restrictions of Φ onto subspace: E_C.

Let's consider a simple example of non-local K-extremum.

<u>4.2. Ex.</u> Returning to the examples above with E^* dual to Banach space E and unit ball $B_1 \subset E^*$, consider the functional in E^*_σ:

$$\Phi(y) = \|y\| \ (y \in B_1), \qquad \Phi(y) = 2 - \|y\| \ (y \notin B_1).$$

Then, since $\{B_1\}$ is universal compact system in E^*_σ, Φ has strong K-minimum at 0. But every weak neighborhood of 0 in E^*_σ intersects both $B_1 \backslash \{0\}$ and $E^* \backslash 2B_1$, hence Φ hasn't local extremum at 0.

Now, let's bring an example of K-continuous but discontinuous in usual sense at zero functional.

<u>4.3. Ex.</u> For the following integral functional it can be proved both abovementioned properties.

$$\varphi(t) = \pi(6\pi - t) \ (0 \leqslant t < 6\pi), \ \Phi(y) = \int_{2\pi}^{4\pi} \varphi\left(\mid y' ln y' \mid^{1-\delta}\right) dx.$$

<u>4.4. Rem.</u> Note that, in view of remarkable Igor Skrypnik theorem (1973) Φ is twice Frechet differentiable only if the integrand f is purely quadratic in y'. It allows to construct easily the examples of the variational functionals being twice K-differentiable but not twice Frechet differentiable, as follows.

5. Calculus of variations in H^1: pseudoquadraticity and K-analyticity.

Remind the classical well-posedness condition for the variational functional (*) in Sobolev space $H^1[a;b]$ with integrand $f(x,y,z)$ has a form
$f(x,y,z) \leqslant \alpha + \beta \cdot z^2$.

Let's introduce much more general condition of pseudoquadraticity in z.

5.1. Def. Say that f is pseudoquadratic in z if f can be written in the following form: $f(x,y,z) = P(x,y)\,[+Q(x,y)\cdot z] + R(x,y,z)\cdot z^2$, where P and Q are locally bounded in x and y, and R is bounded locally in x and y, and globally in z; denote the class by $K_2(z)$.

The following statement is true.

5.2. Thm. If f belongs to $K_2(z)$ then $\Phi(y)$ is well posed in $H^1[a;b]$.

Next, let's introduce conditions guaranteing K-analytical properties of the variational functionals in $H^1[a;b]$.

5.3. Def. Introduce Weierstrass classes for the dominant coefficient R in the pseudoquadratic representation of f. Note that really we consider the corresponding spaces with dominated mixed smoothness.
(i) R belongs to class $W_K(z)$ if R is uniformly continuous and bounded locally in x and y, and globally in z.
(ii) R belongs to class $W_K^1(z)$ if the same is true for R and $\bigtriangledown R$.
(iii) R belongs to class $W_K^2(z)$ if the same is true for R and $\bigtriangledown R$ and $\bigtriangledown^2 R$.

The following facts take place.

5.4. Thm. (i) Continuity of integrand and belonging R to $W_K(z)$ (i.e., $f \in W_K(z)$) implies K-continuity of Φ.
(ii) C^1-smoothness of integrand and belonging R to $W_K^1(z)$ (i.e. $f \in W_K^2(z)$) implies K-differentiability of Φ.
(iii) C^2-smoothness of integrand and belonging R to $W_K^2(z)$ (i.e. $f \in W_K^2(z)$) implies repeated K-differentiability of Φ.

6. Calculus of variations in H^1: conditions of K-extremum.

Application of the K-analytical characteristics permits to obtain at first K-analogs of the classical analytical extreme conditions. But further it can possible to pass to the usual derivatives of the integrand and to prove the analogs of the classical local extreme conditions for variational functionals in C^1.

6.1. Thm. *Euler-Lagrange equation*: If f from W_K^1, $f_z \mid_y$ is absolutely continuous and Φ has K-extremum at y, then a generalized Euler-Lagrange equation takes place (i.e., y is K-extremal).

$$f_y(x,y,y') - \frac{d}{dx}\left[f_z(x,y,y')\right] = 0 \text{ (a. e.)}$$

6.2. Thm. *Legendre necessary condition.* If f from W_K^2, $f_{yz}\mid_y$ is absolutely continuous Φ has K-extremum at y, then a generalized Legendre inequality takes place.

$$f_{z^2}(x,y,y') \geqslant 0 \text{ (a. e.)}$$

6.3. Thm. *Legendre-Jacobi condition.* Let f from W_K^2, $f_z\mid_y$ and $f_{yz}\mid_y$ are absolute continuous, y is K-extremal of Φ. Suppose that:
(i) the strengthened Legendre condition is fullfilled: $f_{z^2}(x,y,y') > 0$;
(ii) the Jacobi condition is fullfilled:

$$\left(-\frac{d}{dx}\left(f_{z^2}\mid_y\right) + \left[-\frac{d}{dx}\left(f_{yz}\mid_y\right) + f_{y^2}\mid_y\right]\cdot U = 0,\ U(a)=0,\ U'(a)=1\right) \Longrightarrow$$

$$\Longrightarrow \left(U'(x) \neq 0,\ a < x \leqslant b\right).$$

Then Φ attains a strong K-minimum at y.

6.4. Rem. Let's note that checking K-extremum at zero in H^1 gives us much more information than one for local extremum in C^1, because the unit ball in C^1 is only one from the infinite quantity of the compact ellipsoids in H^1. The following condition takes place.

$$\left(\text{K-extr. in } H^1\right) \Longrightarrow \left(\text{local extr. in } C^1\right) \Longrightarrow \left(\text{extr. along «K-direction» in } H^1\right).$$

7. Calculus of variations in H^1: non-local K-extremae.

Indeed, it seems that most of non absolute extrema in H^1 are non local as well, and K-extrema of variational functionals in H^1 play role which is analogous to one for the local extrema in C^1. Let's bring some concrete examples.

7.1. Ex. *«Sobolev cos-norm».* $\Phi(y) = \int\limits_0^T (y'^2 \cos y' + y^2)dx$. Classical Sobolev norm, of course, attains an absolute minimum at zero, but «cos-norm» attains only non-local strong K-minimum at zero.

7.2. Ex. *«Quasiharmonic cos-oscillator».* $\Phi(y) = \int\limits_0^T (y'^2 \cos y' - y^2)dx \quad (T <$

π). Classical harmonic oscillator, as it's well known, doesn't attain an absolute extremum for $T > \pi$ in H^1 and attains it for $T < \pi$.

However «quasiharmonic cos-oscillator» attains strong JK-minimum for $T < \pi$ in H^1, and doesn't attain a local minimum for all positive T.

The examples above can be generalized as below in examples 7.3 and 7.4.

7.3. Ex. *«Sobolev quasinorm».* $\Phi(y) = \int\limits_0^T (y' \cdot \varphi(y')^2 + y^2)dx.$

$$\left(\varphi \in W_K^2, \text{ even }, \varphi(0) > 0, \text{ change of sign}\right).$$

7.4. Ex. *«Quasiharmonic oscillator».* $\Phi(y) = \int\limits_0^T (y' \cdot \varphi(y')^2 - y^2)dx.$

$$\left(\varphi \in W_K^2, \text{ even}, \varphi(0) > 0, \text{ change of sign}, \ T < \pi\sqrt{\varphi(0)}\right).$$

8. K-subdifferentials and Bochner integral.

Now we consider a compact convex property of mappings acting from an interval into LCS being very useful for research of the indefinite Bochner integral. First, let's define a consept of K-limit.

8.1. Def. Let E be a LCS, $B = \{B_t\}_{t \in T}$ be a system of the closed convex subsets of E, projectively ordered opposite to inclusion, B be intersection of B_t and B be compact. Say that B is K-limit of B_t if the following condition of «topological contraction» takes place.

Next, let's define K-subdifferential.

8.2. Def. For F acting from I to E define first partial convex subdifferentials as follows $\partial_{co}F(x,h) = \overline{co}\{F(x+h') - F(x)/h' \mid 0 <\mid h' \mid< h\}$. K-subdifferential of F at x is defined as K-limit of the partial convex subdifferentials.

Note some useful properties of K-subdifferentials, including subadditivity (i) and K-mean value theorem (iv).

8.3. Thm. Properties: (i) $\partial_K(F_1 + F_2) \subset \partial_K F_1 + \partial_K F_2$.
(ii) $(F : I \to E_1, \ A \in L(E_1; E_2)) \Longrightarrow (\partial_K(A \circ F) = A\,(\partial_K F))$
(iii) $\partial_K(F_1, F_2) \subset \partial_K F_1 \times \partial_K F_2)$.
(iv) Mean Value: $\quad (F(b) - F(a)/b - a) \in \overline{co}\ \partial_K F((a; b))$.

Using K-subdifferentials to Bochner integrals allows us to strengthen and to generalize for the Frechet spaces a well known for Banach spaces description of indefinite Bochner integral.

8.4. Thm. Given a Frechet space E and a mapping F from I into E the following conditions are equivalent:
(i) F is indefinite Bochner integral.
(ii) F is strongly absolutely continuous and differentiable a.e. on I.
(iii) F is strongly absolutely continuous and K-subdifferentiable a.e. on I.

9. K-absolute continuity and Bochner integral.

As it was noted, not every absolutely continuous mapping can be represented by indefinite Bochner integral. This is a well known Radon-Nikodym problem. Now we introduce a new type of absolute continuity, namely, K-absolute continuity, connected to K-decomposition of the range of values and much more closely connected to Bochner integral.

9.1. Def. Say that $F : I \to E$ is K-absolutely continuous if such compact $C \in \mathcal{C}(E)$ exists that F maps I into $F(a) + E_C$ and F is strongly absolutely continuous mapping into E_C.

Let's mark some properties of class AC_K.

9.2. Some properties.

(i) $AC_K(I,E) \subset AC(I,E) \quad (\Longleftrightarrow$ for $\dim E < \infty)$.

(ii) (E complete) $\Longrightarrow$ ($AC_K(I,E)$ linear).

(iii) (E — Banach) $\Longrightarrow$ $(AC_K(I,E^*_\sigma) = AC_K(I,E^*))$.

Select especially an important property of the a.e. K-subdifferentiability and, in case of Frechet space E, a.e. differentiability of K-absolutely continuous mappings

9.3. Differentiation.

(i) $(F \in AC_K(I,E)) \Longrightarrow$ (F is K-subdifferentiable a.e.).

(ii) ($F \in AC(I,E)$, E — Frechet) $\Longrightarrow$ (F is differentiable a.e.).

It allows to state connection of the class AC_K to Bochner integral.

9.4. Connection to Bochner integral. Let E be a separable LCS. Then $(F \in AC_K(I,E))$ iff the following two properties are fulfilled.

(i) $F \in W^{1,1}(I,E)$ (indefinite Bochner integral).

(ii) $\int_a^b \|f(t)\|_C dt < \infty$ for some $C \in \mathcal{C}(E)$.

9.5. Rem. It can construct an example with $AC_K \subsetneqq W^{1,1}$ (there E isn't Frechet space). A main question now is the following:

Is it possible in case of Frechet space ?

We are going to obtain a negative answer for this question: in Frechet spaces K-absolute continuous mappings are precisely indefinite Bochner integrals.

10. Limit form of Radon-Nikodym property for Frechet spaces.

The classical Radon-Nikodym property for the space E consists of coincidence of the classes of absolutely continuous mappings $AC(I,E)$ and indefinite Bochner integrals $W^{1,1}(I,E)$. Here we state representability of each Sobolev space $W^{1,1}(I,E)$ with Frechet E in view of topological inductive limit both of the spaces $AC(I,E_C)$ and the spaces $W^{1,1}(I,E_C)$. It solves, in a certain sense, the Radon-Nikodym problem for Frechet spaces.

First, let's introduce exactly the spaces under consideration.

10.1. Def. *Spaces under consideration.* $(E, \{\|\cdot\|_n\}_{n=1}^\infty)$ — Frechet space.

(i) $W^{1,1}(I,E) = \left\{ F(x) = F(a) + (B)\int\limits_a^b F'(t)dt \right\}$,

$\|F\|^n = \|F(a)\|_n + \int\limits_a^b \|F'(t)\|_n dt.$

(ii) $W^{1,1}(I,E_C)$: $\|F\|^C = \|F(a)\|_C + \int\limits_a^b \|F'(t)\|_C dt. \quad (C \in \mathcal{C}(E))$

(iii) $AC(I,E_{C'})$: $\|F\|^{C'} = \|F(a)\|_{C'} + \int\limits_a^b \|F'(t)\|_{C'} dt. \quad (C' \in \mathcal{C}(E))$

(iv) $\overset{\circ}{W}{}^{1,1}(I,E) = \{F \in W^{1,1}(I,E) \mid F(a) = 0\} \cong L^1(I,E) \quad (F \leftrightarrow F')$.

Next, it follows easily from Theorem 9.4 the Radon-Nikodym K-property: coincidence of the classes $AC_K(I,E)$ and $W^{1,1}(I,E)$ for Frechet case.

10.2. Radon-Nikodym K-property.

(E — Frechet space) $\Longrightarrow \left(W^{1,1}(I,E) = AC_K(I,E)\right)$. That is:

$$\forall f \in L^1(I,E)\ \exists C \in \mathcal{C}(E):\ F(x) = F(a) + (B)\int\limits_a^b f(t)dt \in AC(I,E_C).$$

11. Limit form of Radon-Nikodym property: main results.

First of all, let's mark that the spectrum $\{E_C\}_{C\in\mathcal{C}(E)}$ generates a corresponding spectrum of Sobolev spaces.

11.1. Theor. For an arbitrary Frechet space E:
(i) the following system $\{W^{1,1}(I,E_C)\}_{C\in\mathcal{C}(E)}$ forms σ-inductive spectrum of Banach spaces (respectively continuous embeddings).
(ii) Moreover, the following «alternation» property takes place.

From here a main result follows immediately.

11.2 For an arbitrary Frechet space E the Sobolev space $W^{1,1}(I,E)$ can be represented as topological limit in the following two ways.

$$W^{1,1}(I,E) \overset{top}{=} \varinjlim_{C\in\mathcal{C}(E)} W^{1,1}(I,E) \overset{top}{\cong} \varinjlim_{C'\in\mathcal{C}(E)} AC(I,E_{C'})$$

That result can be rewritten for the integral spaces as follows.

11.3. Cor. E — Frechet space:

$$L^1(I,E) \overset{top}{=} \varinjlim_{C\in\mathcal{C}(E)} L^1(I,E) \overset{top}{\cong} \varinjlim_{C'\in\mathcal{C}(E)} \overset{\circ}{AC}(I,E_{C'}).$$

11.4. Rem. Note, in conclusion of the part, that, by virtue of a known inductive limit theorem, continuity of an arbitrary linear operator on $W^{1,1}(I,E)$ is equivalent to continuity of the all its restrictions both to the spaces $W^{1,1}(I,E_C)$ and the spaces $AC(I,E_C)$. That allows to «pass over» absence of the classical Radon-Nikodym property.

12. Convex K-characteristics and K-analog of Banach-Zaretsky theorem.

The classical Banach-Zaretsky theorem describing absolute continuous functions as precisely continuous, having bounded variation and Lusin's null property, is one of the important facts in metric theory of functions. Here we'll introduce the convex compact analogs of the main concept of the function theory. It allows us to obtain a precise K-analog of Banach-Zaretsky theorem.

Note that an important K-characteristic, K-subdifferential, closely connected to convex K-absolute continuity, was considered by us earlier.

First, let's introduce an auxiliary notion that permit's to exclude further vector series.

12.1. Def. Let E be vector space, and the subsets $A_k \subset E$ ($k \in \mathbb{N}$) be absolutely convex. Denote by $\bigoplus\limits_{k=1}^{\infty} A_k$ the following set $\{a_{k_1}+...+a_{k_n} \mid a_{k_i}\in A_{k_i},\ i=\overline{1,n};\ n\in\mathbb{N}\}$. Besides, introduce oscillation of A in LCS as follows $\omega(A)=\overline{co}\,(A-A)$.

Now, let's introduce a convex K-absolute continuity by the following way.

12.2. *Convex K-absolute continuity.* $F\in AC_K^{co}(I,E) :\Longleftrightarrow \exists C\in\mathcal{C}(E)$ $\forall\varepsilon>0\ \exists\delta>0$

$$\left(S=\bigcup_k I_k\subset I,\ \sum_k \mid I_k\mid<\delta\right)\Longrightarrow\left(sup\left\|\bigoplus_k \omega F(I_k)\right\|_C<\varepsilon\right).$$

In analogous way the classes of the convex absolutely continuous mappings $AC^{co}(I,E)$ and convex boundedly absolutely continuous mappings $AC_B^{co}(I,E)$ can be considered.

Let's select some properties of the class $AC_K^{co}(I,E)$.

12.3. *Some properties.*

(i) $AC_K(I,E) \subset AC^{co}(I,E) \subset AC^w(I,E)$.

(ii) *Class* $AC_K^{co}(I,E)$ *– linear.*

(iii) $(F_i \in AC_K^{co}(I,E_i) \Longrightarrow ((F_1,F_2) \in AC_K^{co}(I,E_1 \times E_2).))$.

(iv) $(F \in AC_K^{co}(I,E_1),\ A \in L_{Hom}(E_1;E_2)) \Longrightarrow (A \circ F \in AC_K^{co}(I,E_2))$.

In analogous way a concept of convex K-variation can be defined.

12.4. *Convex K-variation.* $P = \bigcup_{k=1}^{n} I_k$, $\mathcal{P} = \{P\}$. Partial convex variation: $V^{co}(F,P) = \bigoplus_{k=1}^{n} \omega F(I_k)$.

Complete convex variation: $\overline{V^{co}(F)} = \overline{\bigcup_{P \in \mathcal{P}} V^{co}(F,P)}$.

$$F \in V_K^{co}(I,E) \iff V^{co}(F) \text{ is compact.}$$

The following statement can be considered as a first step to the K-analog of Banach-Zaretsky theorem:

12.5. Thm. Each convex K-absolute continuous mapping possesses convex K-variation.

13. Convex K-null measure and convex Lusin's K-null property.

Let's pass to the following K-properties. First, let's introduce a concept of convex compact null measure in LCS.

13.1. Def. Say that a subset D from E has a convex K-null measure if for some $C \in \mathcal{C}(E)$ the following condition's are fullfiled:

$$mes_K^{co}(D) = 0 :\iff \exists C \in \mathcal{C}(E)\ \forall \varepsilon > 0\ \exists \bigcup_k U_k \supset D.$$

Here $\{U_k\}$ is an open covering of D.

13.2. Properties.

(i) $mes_K^{co}(D) = 0 \Longrightarrow mes^w(D) = 0$ ($\dim E < \infty$: $\iff$).

(ii) $\{D \subset E \mid mes_K^{co}(D)\}$ *– complete σ-ring.*

(iii) $(D_i \subset E_i,\ mes_K^{co}(D) = 0) \Longrightarrow (mes_K^{co}(D_1 \times D_2) = 0$ in $E_1 \times E_2)$.

Note a connection of the convex K-null measure to the weak one and some other properties of the class. Now, let's introduce a convex compact analog of Lusin N-property.

13.3. Say that a mapping F possesses convex Lusin's K-null property if F transfers each set of a usual null measure into a set of convex K-null measure.

Note the each mapping from AC_K^{co} possesses Lusin's K-null property.

13.4. Connection N_K^{co} to AC_K^{co}: $AC_K^{co}(I,E) \subset N_K^{co}(I,E)$.

As a corollary, the «right part» of Banach-Zaretsky K-theorem follows.

Coroll.: $AC_K^{co}(I,E) \subset C(I,E) \bigcap V_K^{co}(I,E) \bigcap N_K^{co}(I,E)$.

Let's select a simple subclass of the class AC_K^{co} with the help of a convex K-Lipshitz-condition.

13.5. *A simple subclass of AC_K^{co}: convex K-Lipschitz condition.*

$$F \in Lip_K^{co}(I,E) \ :\Longleftrightarrow \ \exists C \in \mathcal{C}(E): \left\| \frac{F(x_2)-F(x_1)}{x_2-x_1} \right\|_C \leqslant M$$

$$\Longleftrightarrow \ F \in Lip(I, E_C).$$

Note the following relation between the classes C^1, Lip_K^{co} and AC_K^{co}.

13.6. Thm. $C^1(I,E) \subset Lip_K^{co}(I,E) \subset AC_K^{co}(I,E)$.

It permits to construct easily the examples of the sets having K-null measure and the mappings from AC_K^{co}.

14. Main results of the item.

First, the convex K-Lipschits mappings into Frechet spaces are precisely the indefinite Bochner integrals from the totally bounded mappings.

14.1. *Description of Lip_K^{co} by Bochner integral.*

E — Frechet space, $F: I \to E$. Then $F \in Lip_K^{co}(I,E) \Longleftrightarrow$

$$F(x) = F(a) + (B)\int_a^x f(t)dt, \quad f \text{ is totally bounded.}$$

Secondly, a precise convex compact analog of Banach-Zaretsky theorem takes place for the mappings into Frechet spaces: the convex K-absolute continuity means precisely continuity, convex compact variation and convex Lusin's K-null property:

14.2. *K-analog of Banach-Zaretsky theorem.* E — Frechet space.

$$AC_K^{co}(I,E) = C(I,E) \bigcap V_K^{co}(I,E) \bigcap N_K^{co}(I,E).$$

Note in the conclusion that in case of the «simply» convex absolute continuous mappings (without compactness) the following «splitting» of the result takes place.

14.3. *«Splitting» for the class AC^{co} (without compactness)*:

$$C(I,E)\bigcap V_K^{co}(I,E)\bigcap N^{co}(I,E) \subsetneqq AC_K^{co}(I,E) \subsetneqq$$

$$\subsetneqq C(I,E)\bigcap V_B^{co}(I,E)\bigcap N^{co}(I,E).$$

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