Direct and inverse theorems of rational approximation in Bergman space

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Bergman space

m_2 — Lebesgue planar measure.
$\Pi := \{z : \mathrm{Im}\, z > 0\}$.

Bergman space

$A_{p,\mu} := A_{p,\mu}(\Pi)$ — Bergman space of analytic in Π functions f for which finite is quasinorm

$$\|f\|_{A_{p,\mu}} = \left(\int_{\Pi} (\mathrm{Im}\, z)^{p\mu-1} |f(z)|^p dm_2(z)\right)^{1/p}, \ p > 0,\ \mu > 0.$$

Besov space

$B_\tau^\alpha, 0 < \tau < \infty,\ \alpha \in \mathbb{R},$ – Besov space of analytic function f in Π, satisfying the condition

$$\|f\|_{B_\tau^\alpha} = \left\|f^{(s)}\right\|_{A_{\tau,s-\alpha}}, \ s = [\alpha] + 1.$$

$f^{(0)} = f$,
$f^{(s)}$ at $s \geq 1$ is s-th derivative f,
$f^{(s)}$ at $s < 0$ is $(-s)$-th antiderivative f, definitely defined by condition $f^{(s)}(z) \to 0$ at $\operatorname{Im} z \to +\infty$.
$\alpha \geq 0$ $\|\cdot\|_{B_\tau^\alpha}$ — half-quasinorm, as $\|f\|_{B_\tau^\alpha} = 0 \Leftrightarrow f$ is algebraic polinomial $\deg f \leq [\alpha]$.
$\|\cdot\|_{B_\tau^\alpha}$, $\alpha \in \left[0, \frac{1}{\tau}\right)$, —- quasinorm if $f(z) \to 0$ at $\operatorname{Im} z \to +\infty$.

The best rational approximation

For spaces B_τ^α, the following continuous noncompact embedding takes place

$$B_{\tau_1}^{\alpha_1} \subset B_{\tau_0}^{\alpha_0} \quad \text{at} \quad \alpha_1 - \alpha_0 = \frac{1}{\tau_1} - \frac{1}{\tau_0} > 0.$$

In particular,

$$B_\tau^\alpha \subset A_{p,\mu} \quad \text{at} \quad \alpha + \mu = \frac{1}{\tau} - \frac{1}{p} > 0.$$

$\mathcal{R}_n$ — set of rational functions with power not higher than n.

The best rational approximation.

$$R_n\left(f\right)_{p,\mu} := R_n\left(f\right)_{A_{p,\mu}} = \inf\left\{\|f - r\|_{A_{p,\mu}} : r \in \mathcal{R}_n \cap A_{p,\mu}\right\}, \quad f \in A_{p,\mu}.$$

$$f \in B_\tau^\alpha \Rightarrow R_n\left(f\right)_{p,\mu} \to 0,\ n \to \infty.$$

Direct and inverse theorems

Theorem

Let positive numbers p, μ, τ and real number α are such that $\alpha + \mu = \frac{1}{\tau} - \frac{1}{p} > 0$ and $\frac{1}{p} + \mu \notin \mathbb{N}$. Then function $f \in A_{p,\mu}$ satisfies the condition

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(n^{\alpha+\mu} R_n (f)_{p,\mu} \right)^{\tau} < \infty$$

in that, and only in that case, when $f \in B_{\tau}^{\alpha}$.

Analogue of theorem for disk was obtained earlier:

$\frac{1}{p} + \mu < 1$ by E. Dyn'kin,
$\mu = \frac{1}{p}, p > 2$, (inverse theorem) by V.R. Misiuk.

Proof of theorem and Bernstein type inequality

Proof of direct theorem is based on atomic representation of functions from Bergman space introduced by R. R. Coifman and R. Rochberg.
Inverse theorem is proved by S. N. Bernstein method for proving inverse theorems of approximation theory and in our case is based on below given Bernstein type inequality

Theorem

Let p and μ are positive numbers so that $\frac{1}{p} + \mu \notin \mathbb{N}$. Then for $\alpha > -\mu$, $\frac{1}{\tau} = \alpha + \mu + \frac{1}{p}$ and $r \in \mathcal{R}_n \cap A_{p,\mu}$, $n \geq 1$, inequality is performed

$$\|r\|_{B_\tau^\alpha} \leq c n^{\alpha+\mu} \|r\|_{A_{p,\mu}}, \quad c = c(p, \mu, \alpha) > 0.$$

Lebesgue spaces

Lebesgue space.

$L_p(\mathbb{R})$ — Lebesgue space of complex-valued functions f on $\mathbb{R}$ with finite quasinorm

$$\|f\|_{L_p(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^p dx \right)^{1/p}, \ 0 < p < \infty.$$

Lebesgue space.

$L_{p,\mu}(\Pi)$ — space of function f complex-valued locally measurable in regard to Lebesgue planar measure m_2 in half-plane Π for which

$$\|f\|_{L_{p,\mu}} = \left(\int_{\Pi} (\mathrm{Im}\, z)^{p\mu-1} |f(z)|^p dm_2(z) \right)^{1/p} < \infty, \ p > 0, \ \mu > 0.$$

$$A_{p,\mu} \subset L_{p,\mu}.$$

Inequality connecting quasinorms rational function in regard to linear and planar measures

Theorem

Let p and μ are positive numbers, such that $\frac{1}{\lambda} = \frac{1}{p} + \mu \notin \mathbb{N}$. Then for $r \in \mathcal{R}_n \cap L_{p,\mu}(\Pi)$, $n \geq 1$, inequality is performed

$$\|r\|_{L_\lambda(\mathbb{R})} \leq cn^{\mu} \|r\|_{L_{p,\mu}(\Pi)}, \quad c = c(p, \mu) > 0.$$

Analogue of theorem for disk was obtained earlier:

$\frac{1}{p} + \mu < 1$ by E. Dyn'kin,
$\mu = \frac{1}{p}$, $p > 2$, by V.R. Misiuk.

Hardy space.

H_p — space of analytical in upper half-plane Π function f for which finite is quasinorm

$$\|f\|_{H_p} := \sup_{y>0} \|f(\cdot + iy)\|_{L_p(\mathbb{R})}.$$

It is known, that if $f \in H_p$, then at almost every point $x \in \mathbb{R}$ there is a $\lim f(z) =: f(x)$, when z tends to x along nontangents to $\mathbb{R}$ paths.

$$\|f\|_{H_p} = \|f\|_{L_p(\mathbb{R})}.$$

Theorem

Let p and μ are positive numbers, such that $\frac{1}{\lambda} = \frac{1}{p} + \mu$. Then for $f \in H_\lambda$ inequality is performed

$$\|f\|_{A_{p,\mu}(\Pi)} \leqslant c\,\|f\|_{H_\lambda}.$$

THANK YOU FOR YOUR ATTENTION!