Entropy numbers of embeddings in Besov spaces

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Let $T \in \mathcal{L}(X, Y)$ be a linear and bounded operator:

$$e_k(T) := \inf \left\{ \varepsilon : \varepsilon > 0 \;\text{ and }\; T(U_X) \subset \bigcup_{j=1}^{2^{k-1}} \{y_j + \varepsilon U_Y\} \right\}$$

$U_X = \{x : \; ||x||_X \leq 1\}$ denotes the unit ball in X .

In particular, T is compact if and only if

$$\lim_{k \to \infty} e_k(T : \; X \to Y) = 0 .$$

Properties: Let $T, R \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, Z)$:

(i) $||T|| = e_1(T) \geq e_2(T) \geq e_3(T) \; . \; . \; . \; \geq 0$

(ii) $e_{k+\ell-1}(T + R) \leq e_k(T) + e_\ell(R)$

(iii) $e_{k+\ell-1}(ST) \leq e_k(S) \cdot e_\ell(T)$

Let X be (quasi-)Banach space and $P: X \to X$ a linear and compact operator.
Let $(\lambda_k)_{k=1}^{\infty}$ be the sequence of all non-zero eigenvalues of P, repeated according to algebraic multiplicity and ordered so that $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq 0$. Then Carl's inequality

$$\lambda_k(P: X \to X) \leq \sqrt{2}\, e_k(P: X \to X)$$

links the behavior of entropy numbers and eigenvalues.

Sometimes it is enough to estimate the entropy numbers of embeddings, for example of the type

$$e_k(id : B^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow B^{s_2}_{p_2,q_2}(\Omega))$$

$$e_k(id : B^{s_1}_{p_1,q_1}(\mathbb{R}^n, < x >^{\alpha_1}) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n, < x >^{\alpha_2}))$$

$$e_k(id : B^{s_1,mloc}_{p_1,q_1}(\mathbb{R}^n, w_1) \hookrightarrow B^{s_2,mloc}_{p_2,q_2}(\mathbb{R}^n, w_2))$$

Useful tools to prove the asymptotic behavior of the entropy numbers of compact embeddings in Besov spaces:

- Wavelet characterization of function spaces
- Nested and weighted sequence spaces
- Entropy numbers of finite-dimensional sequence spaces
- Operator ideals

Classical Fourier-analytical approach for Besov spaces

Let $\varphi_0 \in S(\mathbb{R}^n)$ with $\varphi_0(\xi) = 1$ for $|\xi| \leq 1$ and $\operatorname{supp} \varphi_0 \subset \{\xi : |\xi| \leq 2\}$.

For $j \geq 1$ define

$$\varphi_j(\xi) := \varphi_0(2^{-j}\xi) - \varphi_0(2^{-j+1}\xi) \quad .$$

$$f = \varphi_0(D)f + \sum_{j=1}^{\infty} \varphi_j(D)f \qquad f \in S'(\mathbb{R}^n)$$

where

$$\varphi_j(D)f(x) := (2\pi)^{-n} \int \int e^{i\xi(x-y)} \varphi_j(\xi) f(y)\, dy d\xi \quad .$$

Different types of Besov spaces:

$$B^s_{p,q}(\mathbb{R}^n) \quad : \quad ||f|B^s_{p,q}|| = \Big(\sum_{j=0}^{\infty} 2^{jsq}||\varphi_j(D)f|L_p||^q\Big)^{1/q}$$

$$B^s_{p,q}(\mathbb{R}^n, w) \quad : \quad ||f|B^s_{p,q}|| = \Big(\sum_{j=0}^{\infty} 2^{jsq}||\varphi_j(D)f|L_p(\mathbb{R}^n, w)||^q\Big)^{1/q}$$

$$w(x) = < x >^\alpha \ \text{ or more general } \quad \varphi(|x|)$$

$$B^\beta_{p,q}(\mathbb{R}^n) \quad : \quad ||f|B^\beta_{p,q}|| = \Big(\sum_{j=0}^{\infty} \beta_j||\varphi_j(D)f|L_p||^q\Big)^{1/q}$$

$$(\beta_j)_j \quad \text{admissible sequence} \quad d_0\beta_j \leq \beta_{j+1} \leq d_1\beta_j$$

$$B^s_{p,q}(\mathbb{R}^n, a) \quad : \quad ||f|B^s_{p,q}(a)|| = \Big(\sum_{j=0}^{\infty} 2^{jsq}||(1+2^j|x|)^a\varphi_j(D)f|L_p||^q\Big)^{1/q}$$

$$B^{s,mloc}_{p,q}(\mathbb{R}^n, w) = \{f \in S'(\mathbb{R}^n) : \; ||f|B^{s,mloc}_{pq}(\mathbb{R}^n, w)||_\varphi < \infty\}$$

$$||f|B^{s,mloc}_{p,q}(\mathbb{R}^n, w)||_\varphi = \left(\sum_{j=0}^{\infty} 2^{jsq}||w_j\varphi_j(D)f|L_p(\mathbb{R}^n)||^q\right)^{1/q}$$

Examples:

$$\begin{aligned}
w_j(x) &= 1 \\
w_j(x) &= (1+|x|)^\alpha \quad \text{or} \quad \varphi(|x|) \;\; \text{admissible function} \\
w_j(x) &= \beta_j \quad \text{admissible sequence} \quad d_0\beta_j \le \beta_{j+1} \le d_1\beta_j \\
w_j(x) &= (1+2^j|x-x_0|)^{s'} \\
w_j(x) &= (1+2^j \mathrm{dist}(x,U))^{s'} \qquad \text{with} \quad U \subset \mathbb{R}^n
\end{aligned}$$

Wavelet decomposition

Let $\psi_M \in C^k(\mathbb{R})$ and $\psi_F \in C^k(\mathbb{R})$ are real compactly supported Dauechies wavelets with

$$\int_{\mathbb{R}} x^{\beta} \psi_M(x)\, dx = 0 \qquad \text{for } \; |\beta| < k \; .$$

By a tensor product procedure these can be generalized to the n-dimensional case.
Let $G = (G_1, \cdots, G_n) \in \; G^j = \{F, M\}^{n^*}$ where n^* indicates that at least one of the components of G must be an M and let $G^0 = \{F, M\}^n$. Define

$$\psi^j_{G,m}(x) := 2^{j\frac{n}{2}} \prod_{r=1}^{n} \psi_{G_r}(2^j x_r - m_r)$$

where $j \in \mathbb{N}_0$, $G \in G^j$ and $m \in \mathbb{Z}^n$ and $x = (x_1, ...x_n)$.

$\{\psi^j_{G,m} : j \in \mathbb{N}_0, G \in G^j, m \in \mathbb{Z}^n\}$ is an orthonormal basis in $L_2(\mathbb{R}^n)$.

Let

$$\{\psi^j_{G,m}(x) : j \in \mathbb{N}_0, G \in G^j, m \in \mathbb{Z}^n\}$$

be this wavelet basis (then $\# G^j = 2^n - 1\,, j \geq 1$),

$$\lambda^j_{G,m}(f) := 2^{j\frac{n}{2}}(f, \psi^j_{G,m}) = 2^{j\frac{n}{2}} \int f(x)\, \psi^j_{G,m}(x)\, dx$$

and

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda^j_{G,m}(f)\, 2^{-j\frac{n}{2}} \psi^j_{G,m} \quad .$$

$f \in B^{s,mloc}_{p,q}(\mathbb{R}^n, w)$ if and only if $\lambda = (\lambda^j_{G,m})_{j,G,m} \in \tilde{b}^{s,mloc}_{p,q}(w)$.

$$||\lambda|\tilde{b}^{s,mloc}_{p,q}(w)|| =$$

$$\Big(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \sum_{G \in G^j} \Big(\sum_{m \in \mathbb{Z}^n} |\lambda^j_{G,m}|^p\, w^p_j(2^{-j}m)\Big)^{q/p}\Big)^{1/q}$$

$$I : f \to \left(2^{j\frac{n}{2}} < f, \psi^j_{G,m} >\right)_{j,G,m}$$

is an isomorphic map from $B^{s,mloc}_{p,q}(\mathbb{R}^n, w)$ onto $\tilde{b}^{s,mloc}_{p,q}(w)$.

$$\begin{array}{ccc} B^{s_1,mloc}_{p_1,q_1}(\mathbb{R}^n, w_1) & \xrightarrow{I} & \tilde{b}^{s_1,mloc}_{p_1,q_1}(w_1) \\ \big\downarrow id & & \big\downarrow id_{seq} \\ B^{s_2,mloc}_{p_2,q_2}(\mathbb{R}^n, w_2) & \xleftarrow{I^{-1}} & \tilde{b}^{s_2,mloc}_{p_2,q_2}(w_2) \end{array}$$

Weighted sequence spaces

$$||\lambda|\ell_q(2^{j\sigma}\ell_p(w))|| := \Big(\sum_{j=0}^{\infty} 2^{j\sigma q}\Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{j,m}|^p w_j^p(2^{-j}m)\Big)^{q/p}\Big)^{1/q}$$

Then

$$\tilde{b}^{s_1,mloc}_{p_1,q_1}(w_1) \quad \hookrightarrow \quad \tilde{b}^{s_2,mloc}_{p_2,q_2}(w_2)$$

is equivalent to

$$\ell_{q_1}\Big(2^{j(s_1-\frac{n}{p_1})}\,\ell_{p_1}(w_1)\Big) \hookrightarrow \ell_{q_2}\Big(2^{j(s_2-\frac{n}{p_2})}\,\ell_{p_2}(w_2)\Big)$$

is equivalent to

$$\ell_{q_1}\Big(2^{j\delta}\,\ell_{p_1}(w)\Big) \hookrightarrow \ell_{q_2}\big(\ell_{p_2}\big)$$

$\delta := (s_1 - \frac{n}{p_1}) - (s_2 - \frac{n}{p_2})$ and $w := w_1/w_2$.

$$||\lambda|\ell_q(\beta_j \ell_p(w))|| := \Big(\sum_{j=0}^{\infty} \beta_j \Big(\sum_{m \in \mathbb{Z}^n} |\lambda_{j,m}|^p w_j^p(2^{-j}m) \Big)^{q/p} \Big)^{1/q}$$

Then

$$\tilde{b}^{\gamma_1, mloc}_{p_1,q_1}(w_1) \hookrightarrow \tilde{b}^{\gamma_2, mloc}_{p_2,q_2}(w_2)$$

is equivalent to

$$\ell_{q_1}\Big(\gamma_1 2^{-j\frac{n}{p_1}} \ell_{p_1}(w_1)\Big) \hookrightarrow \ell_{q_2}\Big(\gamma_2 2^{-j\frac{n}{p_2}} \ell_{p_2}(w_2)\Big)$$

is equivalent to

$$\ell_{q_1}\Big(\beta_j \, \ell_{p_1}(w)\Big) \hookrightarrow \ell_{q_2}(\ell_{p_2})$$

$\beta_j := \gamma_{1,j} \, \gamma_{2,j}^{-1} \, 2^{-j(\frac{n}{p_1} - \frac{n}{p_2})}$ and $w := w_1/w_2$.

Theorem: $\ell_{q_1}\Big(\beta_j\,\ell_{p_1}(w)\Big) \hookrightarrow \ell_{q_2}(\ell_{p_2})$ holds if and only if

$$\Big(\beta_j^{-1}\,\|\,((w_{j,m})^{-1})_m\,|\ell_{p*}\|\Big)_j \in \ell_{q*}$$

where

$$\frac{1}{p^*} := \left(\frac{1}{p_2} - \frac{1}{p_1}\right)_+ \qquad \text{and} \qquad \frac{1}{q^*} := \left(\frac{1}{q_2} - \frac{1}{q_1}\right)_+ .$$

The embedding is compact if and only if in addition

$$\lim_{j\to\infty} \beta_j^{-1}\,\|\,((w_{j,m})^{-1})_m\,|\ell_{p*}\| = 0 \qquad \text{if} \quad q^* = \infty\,,$$

and

$$\lim_{|m|\to\infty} w_{j,m} = \infty \qquad \text{for all } \ j \in \mathbb{N}_0 \qquad \text{if} \quad p^* = \infty\,.$$

Here $w_{j,m} := w_j(2^{-j}m)$.

Corollary: Let $w_j(x) =< x >^{\alpha}$, that is $w_{j,m} = (1 + |2^{-j}m|^2)^{\alpha/2}$. Then the embedding

$$\ell_{q_1}\left(2^{j\delta}\,\ell_{p_1}(w)\right) \hookrightarrow \ell_{q_2}(\ell_{p_2})$$

is compact if and only if $\min(\delta, \alpha) > n/p*$.

If we consider $w_j(x) \sim \varphi(|x|)$ then the embedding is compact if and only if $\delta > n/p*$ and $\int_1^{\infty} \varphi(s)^{-p*} s^n \frac{ds}{s} < \infty$ in case $0 < p* < \infty$.

$\varphi : [1, \infty) \to (0, \infty)$ is admissible if it is positive, measurable, and satisfies

$$0 < \underline{\varphi}(t) := \inf_{s \in [1,\infty)} \frac{\varphi(ts)}{\varphi(s)} \quad , \quad \overline{\varphi}(t) := \sup_{s \in [1,\infty)} \frac{\varphi(ts)}{\varphi(s)} < \infty$$

Let $w_{\ell,j}(x) = (1 + 2^j \text{dist}(x, U))^{s'_\ell} \quad \ell = 1\,,\, 2\,.$ Consequently we have $w_{j,m} = (1 + 2^j\, \text{dist}(2^{-j}m, U)^{s'}$ where $s' = s'_1 - s'_2$.

Lemma:

(i) If U is unbounded then the embedding

$$B^{s_1,mloc}_{p_1,q_1}(\mathbb{R}^n, w_1) \hookrightarrow B^{s_2,mloc}_{p_2,q_2}(\mathbb{R}^n, w_2)$$

can never become compact.

(ii) Let U be a d-set. Then the embedding

$$B^{s_1,mloc}_{p_1,q_1}(\mathbb{R}^n, w_1) \hookrightarrow B^{s_2,mloc}_{p_2,q_2}(\mathbb{R}^n, w_2)$$

is compact if and only if

$$s' > n/p* \quad \text{and} \quad \delta > d/p* \quad .$$

Let U be a compact set and μ a Radon measure with supp $\mu = U$.
Let $\gamma \in U$ and $0 < r < 1$.
U is called a d-set if $\mu(B(\gamma, r)) \sim r^d\,, \quad 0 \le d \le n\,.$

j \ m							
	$I_{0,0}$	$I_{0,1}$	$I_{0,2}$			$I_{0,M}$	
	$I_{1,0}$	$I_{1,1}$			$I_{1,M-1}$		
				$I_{j,i}$			
	$I_{M,0}$				$I_{M,M-1}$		

We introduce a certain decomposition of the identity

$$\Lambda := \{\lambda = (\lambda_{j,m})_{j\in\mathbb{N}_0,\, m\in\mathbb{Z}^n} : \quad \lambda_{j,m} \in \mathbb{C}, \quad j \in \mathbb{N}_0,\, m \in \mathbb{Z}^n\}\,.$$

Let $I_{j,i} \subset \mathbb{N}_0 \times \mathbb{Z}^n$ s.t.

$$\begin{aligned} I_{j,i} &:= \{(j,m) : \qquad\qquad\qquad\qquad \}\,,\ i \in \mathbb{N}\,,\ j \in \mathbb{N}_0 \\ I_{j,0} &:= \{(j,m) : \qquad\qquad\qquad\qquad \}\,, \qquad j \in \mathbb{N}_0\,. \end{aligned}$$

Further, let $P_{j,i} : \Lambda \to \Lambda$ be the canonical projection with respect to $I_{j,i}$, i.e., for $\lambda \in \Lambda$ we put

$$(P_{j,i}\lambda)_{u,v} := \begin{cases} \lambda_{u,v} & (u,v) \in I_{j,i}\,, \\ 0 & \text{otherwise.} \end{cases}\,, \qquad u \in \mathbb{N}_0, \quad v \in \mathbb{Z}^n\,.$$

$$\mathrm{id}_\Lambda = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} P_{j,i}\,.$$

$$\Lambda := \{\lambda = (\lambda_{j,m})_{j\in\mathbb{N}_0,\, m\in\mathbb{Z}^n} : \quad \lambda_{j,m}\in\mathbb{C}, \quad j\in\mathbb{N}_0,\, m\in\mathbb{Z}^n\}\,.$$

Let $w_j(x) = < x >^{\alpha}$

$$\begin{aligned} I_{j,i} &:= \{(j,m) :\ 2^{j+i-1} < |m| \le 2^{j+i}\}\,,\ i\in\mathbb{N}\,,\ j\in\mathbb{N}_0 \\ I_{j,0} &:= \{(j,m) : \quad |m| \le 2^{j}\}\,, \qquad\qquad j\in\mathbb{N}_0\,. \end{aligned}$$

Further, let $P_{j,i} : \Lambda \to \Lambda$ be again the canonical projection with respect to $I_{j,i}$.

We have $w_j(2^{-j}m) = (1 + (2^{-j}|m|)^2)^{\alpha/2} \sim 2^{i\alpha}$ and the cardinality of $I_{j,i}$ denoted by $M_{j,i} \sim 2^{(j+i)n}$.

$$e_k(P_{j,i} : \ell_{q_1}(2^{j\delta}\,\ell_{p_1}(w)) \hookrightarrow \ell_{q_2}(\ell_{p_2})) \le 2^{-j\delta}2^{-i\alpha} e_k(id : \ell_{p_1}^{M_{j,i}} \to \ell_{p_2}^{M_{j,i}})$$

$$\Lambda := \{\lambda = (\lambda_{j,m})_{j\in\mathbb{N}_0,\, m\in\mathbb{Z}^n} : \quad \lambda_{j,m} \in \mathbb{C}, \quad j \in \mathbb{N}_0,\, m \in \mathbb{Z}^n\}\,.$$

Let $w_j(x) = (1 + 2^j \mathrm{dist}(x, U))^{s'}$

$$I_{j,i} := \{(j,m) :\ \sqrt{n}\, 2^{-j+i-1} < \mathrm{dist}(2^{-j} m, U) \le \sqrt{n}\, 2^{-j+i}\}\,,$$

$$I_{j,0} := \{(j,m) :\ \mathrm{dist}(2^{-j} m, U) \le \sqrt{n}\, 2^{-j}\}\,, \qquad i \in \mathbb{N}\,,\, j \in \mathbb{N}_0\,.$$

Further, let $P_{j,i} : \Lambda \to \Lambda$ be again the canonical projection with respect to $I_{j,i}$.

We have $w_j(2^{-j} m) \sim (1 + 2^j 2^{-j+i})^{s'} \sim 2^{is'}$ and

$$e_k(P_{j,i} : \ell_{q_1}(2^{j\delta}\, \ell_{p_1}(w)) \hookrightarrow \ell_{q_2}(\ell_{p_2})) \le 2^{-j\delta} 2^{-is'}\, e_k(id : \ell_{p_1}^{M_{j,i}} \to \ell_{p_2}^{M_{j,i}})$$

Lemma: Let U be a d-set, than

$$M_{ji} \sim \begin{cases} 2^{in} 2^{(j-i)d} & 0 \le i < j\,, \\ 2^{in} & j \le i. \end{cases}$$

Operator ideals

Given a bounded linear operator $P \in \mathcal{L}(X, Y)$, where X and Y are quasi-Banach spaces, and a positive real number r we put

$$L_{r,\infty}^{(e)}(P) := \sup_{k \in \mathbb{N}} k^{1/r} \, e_k(P) \, .$$

This is a quasi-norm for the operator ideal $\mathcal{L}_{r,\infty}^{(e)}$.

- $L_{r,\infty}^{(e)}(P) < c$ if and only if $e_k(P) \leq c \, k^{-\frac{1}{r}}$
- There exist an equivalent ϱ-norm on $\mathcal{L}_{r,\infty}^{(e)}$
- $\mathcal{L}_{r,\infty}^{(e)}$ is complete and $\Big[L_{r,\infty}^{(e)}\big(\sum_j P_j\big)\Big]^{\varrho} \leq \sum_j \Big[L_{r,\infty}^{(e)}(P_j)\Big]^{\varrho}$

Entropy numbers in finite-dimensional spaces

Let $0 < p_1 < p_2 \le \infty$

$$e_k\left(\mathrm{id} : \ell_{p_1}^N \to \ell_{p_2}^N\right) \sim \begin{cases} 1 & \text{if} \quad 1 \le k \le \log 2N\,, \\ \left(\frac{\log(1+\frac{N}{k})}{k}\right)^{\frac{1}{p_1}-\frac{1}{p_2}} & \text{if} \quad \log 2N \le k \le 2N\,, \\ 2^{-\frac{k}{2N}}\, N^{\frac{1}{p_2}-\frac{1}{p_1}} & \text{if} \quad 2N \le k\,. \end{cases}$$

and in case $0 < p_2 < p_1 \le \infty$ it holds

$$e_k\left(\mathrm{id} : \ell_{p_1}^N \to \ell_{p_2}^N\right) \sim 2^{-\frac{k}{2N}}\, N^{\frac{1}{p_2}-\frac{1}{p_1}} \qquad \text{for all } k \in \mathbb{N}.$$

This implies

$$L_{r,\infty}^{(e)}(\mathrm{id} : \ell_{p_1}^N \to \ell_{p_2}^N) \;\sim\; N^{\frac{1}{r}-(\frac{1}{p_1}-\frac{1}{p_2})} \qquad \text{if} \quad \frac{1}{r} > \max\left(0, \frac{1}{p_1}-\frac{1}{p_2}\right).$$

Estimates in case $w_\alpha(x) = (1+|x|^2)^{\alpha/2}$

Theorem: *(The nonlimiting case)*
Let $0 < p_1, p_2, q_1, q_2 \le \infty$, $s_2 < s_1$ and $\alpha > 0$.
$\delta = s_1 - s_2 + (\frac{n}{p_2} - \frac{n}{p_1})$.
Let $\min(\delta, \alpha) > \frac{n}{p*}$ and $\delta \ne \alpha$.
Then

$$e_k\big(id : B^{s_1}_{p_1,q_1}(\mathbb{R}^n, < x >^\alpha) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n)\big) \sim k^{-\frac{\min(\delta,\alpha)}{n} - \frac{1}{p_1} + \frac{1}{p_2}} \quad .$$

Remark: Let $1 < p_1,\ p_2\ < \infty$. Then

$$e_k\big(id : H^{s_1}_{p_1}(\mathbb{R}^n, < x >^\alpha) \hookrightarrow H^{s_2}_{p_2}(\mathbb{R}^n)\big) \sim k^{-\frac{\min(\delta,\alpha)}{n} - \frac{1}{p_1} + \frac{1}{p_2}} \quad .$$

It holds

$$e_k(P_{j,i}) \leq c\, 2^{-j\delta - i\alpha}\, e_k(\mathrm{id} : \ell_{p_1}^{M_{j,i}} \to \ell_{p_2}^{M_{j,i}})$$

and consequently

$$L_{r,\infty}^{(e)}(P_{j,i}) \leq c\, 2^{-j\delta - i\alpha}\, L_{r,\infty}^{(e)}(\mathrm{id} : \ell_{p_1}^{M_{j,i}} \to \ell_{p_2}^{M_{j,i}})\,.$$

To shorten notations let $1/p = 1/p_1 - 1/p_2$. Under the assumption $1/r > \max(0, 1/p)$ we have

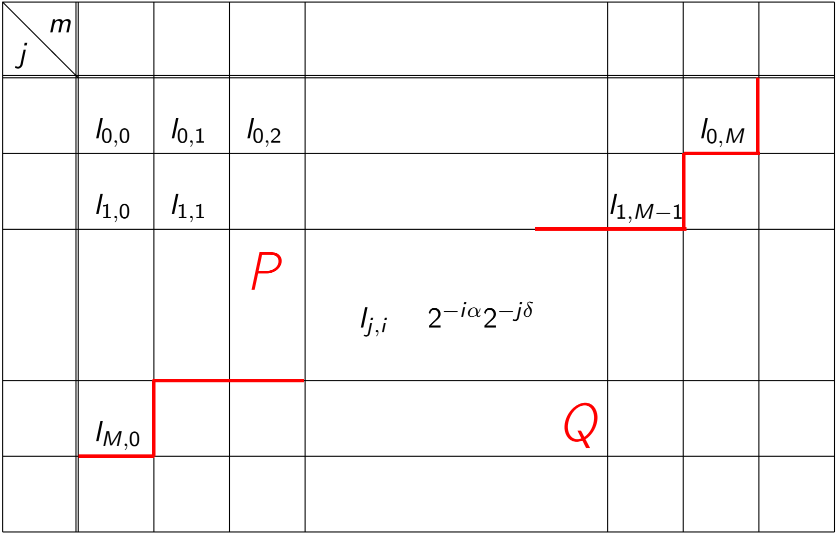
$$L_{r,\infty}^{(e)}(\mathrm{id} : \ell_{p_1}^{M_{j,i}} \to \ell_{p_2}^{M_{j,i}}) \sim 2^{n(j+i)(\frac{1}{r} - \frac{1}{p})}$$

and so it follows

$$L_{r,\infty}^{(e)}(P_{j,i}) \leq c\, 2^{-j\delta - i\alpha}\, 2^{n(j+i)(\frac{1}{r} - \frac{1}{p})}\ .$$

Now, for given $M \in \mathbb{N}_0$ let

$$P := \sum_{m=0}^{M} \sum_{j+i=m} P_{j,i} \quad \text{and} \quad Q := \sum_{m=M+1}^{\infty} \sum_{j+i=m} P_{j,i}\,.$$



$$P := \sum_{m=0}^{M} \sum_{j+i=m} P_{j,i} \quad \text{and} \quad Q := \sum_{m=M+1}^{\infty} \sum_{j+i=m} P_{j,i}$$

Estimate of $L^{(e)}_{r,\infty}(P)$

$$
\begin{aligned}
L^{(e)}_{r,\infty}(P)^{\varrho} &\leq \sum_{m=0}^{M} \sum_{j+i=m} L^{(e)}_{r,\infty}(P_{j,i})^{\varrho} \\
&\leq c_1 \sum_{m=0}^{M} \sum_{j+i=m} 2^{\varrho(-j\delta - i\alpha)} 2^{\varrho n m(\frac{1}{r}-\frac{1}{p})} \\
&\leq c_2 \sum_{m=0}^{M} 2^{\varrho n m(\frac{1}{r}-\frac{1}{p}-\frac{\alpha}{n})} \sum_{j=0}^{m} 2^{\varrho j(\alpha-\delta)} \\
&\leq c_3 \sum_{m=0}^{M} 2^{\varrho n m(\frac{1}{r}-\frac{1}{p}-\frac{\alpha}{n})} \times \begin{cases} 2^{\varrho m(\alpha-\delta)} & \text{if} \quad \alpha > \delta\,, \\ m+1 & \text{if} \quad \alpha = \delta\,, \\ 1 & \text{if} \quad \alpha < \delta\,. \end{cases}
\end{aligned}
$$

Let $\alpha < \delta$. If r is now chosen such that

$$\frac{1}{r} > \max\Big(0, \frac{1}{p}\Big) \quad \text{and} \quad n\Big(\frac{1}{r} - \frac{1}{p}\Big) - \alpha > 0\,,$$

then follows

$$L^{(e)}_{r,\infty}(P) \le c_4\, 2^{nM(\frac{1}{r} - \frac{1}{p} - \frac{\alpha}{n})}$$

and this implies

$$e_{2Mn}(P : \ell_{q_1}(2^{j\delta}\, \ell_{p_1}(w_\alpha)) \to \ell_{q_2}(\,\ell_{p_2}))) \le\; c_4\, 2^{nM(-\frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n})}\,.$$

Now we consider the case $\alpha > \delta$. Then

$$L^{(e)}_{r,\infty}(P)^{\varrho} \;\le\; c_3 \sum_{m=0}^{M} 2^{\varrho nm(\frac{1}{r} - \frac{1}{p} - \frac{\delta}{n})}$$

and similar as above, we find

$$e_{2Mn}(P : \ell_{q_1}(2^{j\delta}\, \ell_{p_1}(w_\alpha)) \to \ell_{q_2}(\,\ell_{p_2}))) \le\; c_4\, 2^{nM(-\frac{1}{p_1} + \frac{1}{p_2} - \frac{\delta}{n})}$$

if

$$\frac{1}{r} > \max\Big(0, \frac{1}{p}\Big) \qquad \text{and} \qquad n\Big(\frac{1}{r} - \frac{1}{p}\Big) - \delta > 0\,.$$

Observe that this is possible if r is chosen small enough.

Estimate of $L^{(e)}_{r,\infty}(Q)$

$$L^{(e)}_{r,\infty}(Q)^{\varrho} \quad \le \quad c_5 \sum_{m=M+1}^{\infty} 2^{\varrho n m(\frac{1}{r}-\frac{1}{p}-\frac{\alpha}{n})} \times \begin{cases} 2^{\varrho m(\alpha-\delta)} & \text{if} \quad \alpha > \delta\,, \\ m+1 & \text{if} \quad \alpha = \delta\,, \\ 1 & \text{if} \quad \alpha < \delta\,. \end{cases}$$

In case $\alpha < \delta$ this leads to

$$L^{(e)}_{r,\infty}(Q) \le c_6\, 2^{nM(\frac{1}{r}-\frac{1}{p}-\frac{\alpha}{n})} \qquad \text{if} \qquad \max\Big(0, \frac{1}{p}\Big) < \frac{1}{r} < \frac{\alpha}{n} + \frac{1}{p}\,.$$

Because of $\alpha > n/p^* = n\, \max(0, -1/p)$ we have

$$\max\Big(0, \frac{1}{p}\Big) < \frac{\alpha}{n} + \frac{1}{p}\,.$$

Hence, there exists an appropriate r. This gives

$$e_{2^{Mn}}(Q : \ell_{q_1}(2^{j\delta}\, \ell_{p_1}(w_\alpha)) \to \ell_{q_2}(\, \ell_{p_2}))) \le \; c_6\, 2^{nM(-\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n})}\,.$$

Using the same type of arguments if $\alpha > \delta$ we derive

$$L_{r,\infty}^{(e)}(Q)^{\varrho} \le \sum_{m=M+1}^{\infty} 2^{\varrho nm(\frac{1}{r}-\frac{1}{p}-\frac{\delta}{n})}$$

and

$$e_{2^{Mn}}(Q : \ell_{q_1}(2^{j\delta}\,\ell_{p_1}(w_\alpha)) \to \ell_{q_2}(\,\ell_{p_2}))) \le \; c_6\, 2^{nM(-\frac{1}{p_1}+\frac{1}{p_2}-\frac{\delta}{n})}$$

where the number r has to be chosen s.t.

$$\max\Big(0, \frac{1}{p}\Big) < \frac{1}{r} < \frac{\delta}{n} + \frac{1}{p}\,,$$

(which is possible because of $\delta > n/p^*$).

Estimate from below

We consider the commutative diagram

$$\begin{array}{ccc} \ell_{p_1}^{M_{j,i}} & \xrightarrow{S_{j,i}} & \ell_{q_1}(2^{j\delta}\ell_{p_1}(w_\alpha)) \\ id_1 \downarrow & & \downarrow id \\ \ell_{p_2}^{M_{j,i}} & \xleftarrow{T_{j,i}} & \ell_{q_2}(\ell_{p_2}) \end{array}$$

Observe

$$\| \, T_{j,i} \, | \big(\ell_{q_2}(\ell_{p_2}) \to \ell_{p_2}^{M_{j,i}} \big) \| \, = \, 1$$

and

$$\| \, S_{j,i} \, | \big(\ell_{p_1}^{M_{j,i}} \to \ell_{q_1}(2^{j\delta}\ell_{p_1}(w_\alpha)) \big) \| \sim 2^{j\delta + i\alpha} \, .$$

$$e_k(\mathrm{id}_1 : \ell_{p_1}^{M_{j,i}} \to \ell_{p_2}^{M_{j,i}})$$

$$\leq \|S_{j,i}\| \, \|T_{j,i}\| \, e_k(\mathrm{id} : \ell_{q_1}(2^{j\delta} \ell_{p_1}(w_\alpha)) \to \ell_{q_2}(\ell_{p_2})))$$

$$\leq c \, 2^{j\delta + i\alpha} \, e_k(\mathrm{id} : \ell_{q_1}(2^{j\delta} \ell_{p_1}(w_\alpha)) \to \ell_{q_2}(\ell_{p_2}))) \, .$$

So we find with $j = 0$ and $k = 2 \cdot 2^{in} \sim 2 \cdot M_{0,i}$

$$2^{in(-\frac{1}{p_1} + \frac{1}{p_2})} 2^{-id\frac{\alpha}{n}} \ \leq \ c \ e_{2.2^{in}}(\mathrm{id} : \ell_{q_1}(2^{j\delta} \ell_{p_1}(w_\alpha)) \to \ell_{q_2}(\ell_{p_2})))$$

and with $i = 0$ and $k = 2 \cdot 2^{jn} \sim 2 \cdot M_{j,0}$

$$2^{jn(-\frac{1}{p_1} + \frac{1}{p_2})} 2^{-jn\frac{\delta}{n}} \ \leq \ c \ e_{2.2^{jn}}(\mathrm{id} : \ell_{q_1}(2^{j\delta} \ell_{p_1}(w_\alpha)) \to \ell_{q_2}(\ell_{p_2}))) \, .$$

Consequently

$$k^{-\frac{\min(\alpha,\delta)}{n} - \frac{1}{p_1} + \frac{1}{p_2}} \ \leq \ c \ e_k(\mathrm{id} : \ell_{q_1}(2^{j\delta} \ell_{p_1}(w_\alpha)) \to \ell_{q_2}(\ell_{p_2}))) \, .$$

We say that the function $\varphi : [1,\infty) \to (0,\infty)$ belongs to $\mathcal{V}$ if φ is positive, measurable, and satisfies

$$0 < \underline{\varphi}(t) := \inf_{s\in[1,\infty)} \frac{\varphi(ts)}{\varphi(s)}$$

$$\overline{\varphi}(t) := \sup_{s\in[1,\infty)} \frac{\varphi(ts)}{\varphi(s)} \quad < \infty$$

$\overline{\varphi}(t)$ is submultiplicative on $[1,\infty)$ and

$$\alpha_\varphi := \inf_{t>1} \frac{\log \overline{\varphi}(t)}{\log t} \qquad \beta_\varphi := \sup_{t>1} \frac{\log \underline{\varphi}(t)}{\log t} \quad .$$

$$\underline{\varphi}(t)\,\varphi(s) \le \varphi(ts) \le \overline{\varphi}(t)\varphi(s)\,.$$

It holds

$$-\infty \,<\, \beta_\varphi \,\le\, \alpha_\varphi \,<\, \infty\,.$$

For any $\varepsilon > 0$ there exists a constant $c_\varepsilon \ge 1$ such that

$$c_\varepsilon^{-1}\, s^{\beta_\varphi-\varepsilon} \le \underline{\varphi}(s) \le \frac{\varphi(s)}{\varphi(1)} \le \overline{\varphi}(s) \le c_\varepsilon\, s^{\alpha_\varphi+\varepsilon}.$$

Theorem: Let $w \in W_1 \cap W_2$. Let $\varphi \in V$ be an associated function, that is $w(x) \sim \varphi(|x|)$.

(i) Suppose $n/p^* < \beta_\varphi \le \alpha_\varphi < \delta$. Then

$$e_k\Big(\mathrm{id} \,:\, B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n)\Big) \sim k^{-(\frac{1}{p_1}-\frac{1}{p_2})}\,(\varphi(k^{1/n}))^{-1}\,.$$

(ii) Suppose $n/p^* < \delta < \beta_\varphi$. Then

$$e_k\big(\mathrm{id} : B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n)\big) \sim k^{-\frac{\delta}{n}-(\frac{1}{p_1}-\frac{1}{p_2})}\,.$$

Kühn, Leopold, Sickel, Skrzypczak:
Entropy numbers of embeddings of weighted Besov spaces II
Proc. Edinburgh Math. Soc. (2), 49, 331-359, 2006.

Let $w_{\ell,j}(x) = (1 + 2^j \mathrm{dist}(x, U))^{s'_\ell}$ $\ell = 1\,,\,2$. Consequently we have $w_{j,m} = (1 + 2^j\, \mathrm{dist}(2^{-j}m, U)^{s'}$ where $s' = s'_1 - s'_2$.

Theorem: Let U be a d-set , $0 \le d \le n$, $\frac{1}{p^*} := \left(\frac{1}{p_2} - \frac{1}{p_1}\right)_+$,

$\delta = s_1 - s_2 - n(1/p_1 - 1/p_2) > d/p*$ and $s' = s'_1 - s'_2 > n/p*$.

Then

$$e_k\left(id : \ell_{q_1}\left(2^{j\delta}\, \ell_{p_1}(w)\right) \hookrightarrow \ell_{q_2}(\ell_{p_2})\right) \sim k^{-\frac{1}{p_1}+\frac{1}{p_2}-\min\left(\frac{\delta}{d},\frac{s'}{n}\right)}$$

and

$$e_k(id : B^{s_1,mloc}_{p_1,q_1}(\mathbb{R}^n, w^{s'_1}) \hookrightarrow B^{s_2,mloc}_{p_2,q_2}(\mathbb{R}^n, w^{s'_2})) \sim k^{-\frac{1}{p_1}+\frac{1}{p_2}-\min\left(\frac{\delta}{d},\frac{s'}{n}\right)} \quad .$$

Skrzypczak and L.: *Entropy numbers of embeddings of some 2-microlocal Besov spaces* *2010*

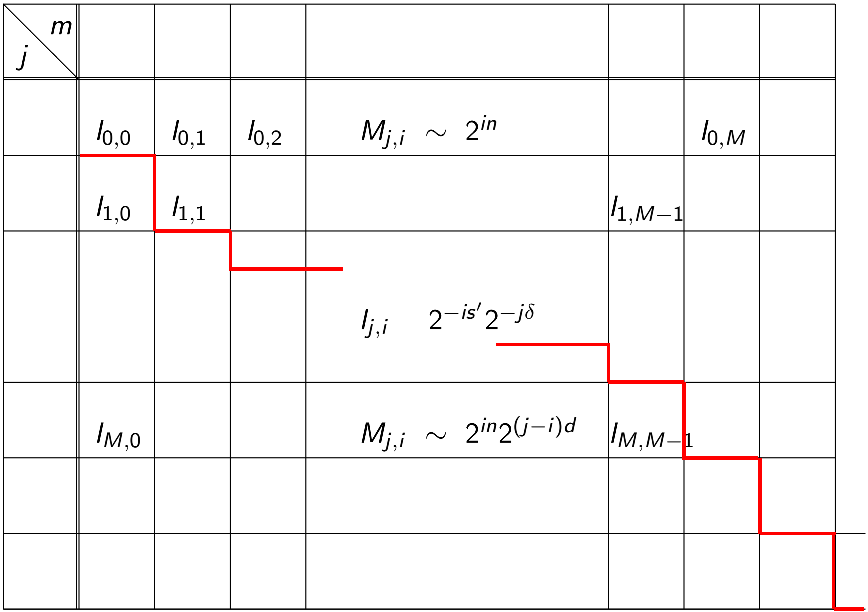
$j \backslash m$							
	$I_{0,0}$	$I_{0,1}$	$I_{0,2}$			$I_{0,M}$	
	$I_{1,0}$	$I_{1,1}$			$I_{1,M-1}$		
				$I_{j,i} \quad 2^{-is'}2^{-j\delta}$			
	$I_{M,0}$				$I_{M,M-1}$		

$$w_j(2^{-j}m) \;\sim\; (1+2^j \,\mathrm{dist}\,(2^{-j}m, U))^{s'} \;\sim\; 2^{is'} \qquad \text{if} \quad (j,m) \in I_{j,i}\,.$$

$$\begin{aligned} e_k(P_{j,i}) &\le \frac{1}{\inf_{m\in I_{j,i}} w_j(2^{-j}m)}\, 2^{-j\delta}\, e_k\Big(\mathrm{id} \,:\, \ell_{p_1}^{M_{j,i}} \to \ell_{p_2}^{M_{j,i}}\Big) \\ &\le c\, 2^{-j\delta} 2^{-is'}\, e_k\Big(\mathrm{id} \,:\, \ell_{p_1}^{M_{j,i}} \to \ell_{p_2}^{M_{j,i}}\Big)\,, \end{aligned}$$

with a constant c independent of k, j and i.
We find

$$L_{r,\infty}^{(e)}(P_{j,i}) \le c\, 2^{-j\delta} 2^{-is'}\, L_{r,\infty}^{(e)}(\mathrm{id} \,:\, \ell_{p_1}^{M_{j,i}} \to \ell_{p_2}^{M_{j,i}})\,.$$



m
j
I0,0
I0,1
I0,2
Mj,i ∼ 2in
I0,M
I1,0
I1,1
I1,M−1
Ij,i 2−is′2−jδ
IM,0
Mj,i ∼ 2in2(j−i)d
IM,M−1

We have

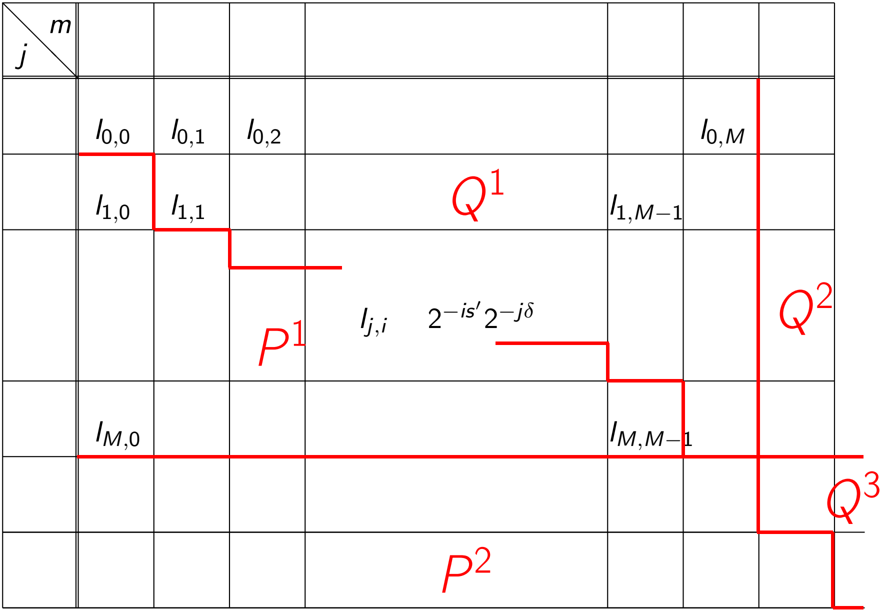
$$L^{(e)}_{r,\infty}(P_{j,i}) \le c\,2^{-j\delta}2^{-is'}\,L^{(e)}_{r,\infty}(\mathrm{id}\,:\,\ell_{p_1}^{M_{j,i}} \to \ell_{p_2}^{M_{j,i}})\,.$$

and

$$M_{ji} \sim \begin{cases} 2^{in}2^{(j-i)d} & 0 \le i < j\,, \\ 2^{in} & j \le i. \end{cases}$$

Now under the assumption $1/r > \max(0, 1/p)$ with $1/p = 1/p_1 - 1/p_2$ we conclude

$$L^{(e)}_{r,\infty}(P_{j,i}) \le c\,2^{-j\delta}2^{-is'} \begin{cases} 2^{(in+d(j-i))(\frac{1}{r}-\frac{1}{p})} \quad , & 0 \le i < j \\ 2^{in(\frac{1}{r}-\frac{1}{p})} \quad , & 0 < j \le i \end{cases}$$


m
j
$I_{0,0}$
$I_{0,1}$
$I_{0,2}$
$I_{0,M}$
Q^1
$I_{1,0}$
$I_{1,1}$
$I_{1,M-1}$
$I_{j,i}$
$2^{-is'}2^{-j\delta}$
P^1
Q^2
$I_{M,0}$
$I_{M,M-1}$
Q^3
P^2

Now, for given $M \in \mathbb{N}_0$ let

$$P^1 := \sum_{j=0}^{M} \sum_{i=0}^{j-1} P_{j,i} \qquad P^2 := \sum_{j=M+1}^{\infty} \sum_{i=0}^{j-1} P_{j,i}$$

$$Q^1 := \sum_{j=0}^{M} \sum_{i=j}^{M} P_{j,i} \qquad Q^2 := \sum_{j=0}^{M} \sum_{i=M+1}^{\infty} P_{j,i}$$

$$Q^3 := \sum_{j=M+1}^{\infty} \sum_{i=j}^{\infty} P_{j,i}$$

Estimate of $L^{(e)}_{r,\infty}(P^1)$.

$$\begin{aligned}
L^{(e)}_{r,\infty}(P^1)^{\varrho} &\le \sum_{j=0}^{M}\sum_{i=0}^{j-1} L^{(e)}_{r,\infty}(P_{j,i})^{\varrho} \\
&\le c_1 \sum_{j=0}^{M}\sum_{i=0}^{j-1} 2^{-j\varrho\delta} 2^{-is'\varrho}\, 2^{(in+d(j-i))\varrho(\frac{1}{r}-\frac{1}{p})} \\
&\le c_2 \sum_{j=0}^{M} 2^{-j\varrho\delta} 2^{\varrho dj(\frac{1}{r}-\frac{1}{p})} \sum_{i=0}^{j-1} 2^{-is'\varrho}\, 2^{i(n-d)\varrho(\frac{1}{r}-\frac{1}{p})} \\
&\le c_3 \sum_{j=0}^{M} 2^{-j\varrho\delta} 2^{\varrho dj(\frac{1}{r}-\frac{1}{p})}\, 2^{-js'\varrho}\, 2^{j(n-d)\varrho(\frac{1}{r}-\frac{1}{p})} \\
&\le c_4\, 2^{-M\varrho\delta}\, 2^{-Ms'\varrho}\, 2^{Mn\varrho(\frac{1}{r}-\frac{1}{p})}
\end{aligned}$$

r is chosen such that

$$\Big(\frac{1}{r}-\frac{1}{p}\Big)(n-d) > s' \qquad \text{and} \qquad d\Big(\frac{1}{r}-\frac{1}{p}\Big) > \delta\,.$$

and gives

$$e_{2^{Mn}}(P^1) \le c_4\, 2^{nM(-\frac{1}{p_1}+\frac{1}{p_2}-\frac{\delta}{n}-\frac{s'}{n})}\,.$$

On the other hand, in case $d=0$, we have to chose r in such a way that again

$$n\Big(\frac{1}{r}-\frac{1}{p}\Big) > s'+\delta$$

holds and obtain the same result.
If we consider the case $d=n$ we obtain in a similar way

$$e_{2^{Mn}}(P^1) \le c_5\, 2^{nM(-\frac{1}{p_1}+\frac{1}{p_2}-\frac{\delta}{d})}\,.$$

Estimate of $L^{(e)}_{r,\infty}(Q^1)$.

$$\begin{aligned} L^{(e)}_{r,\infty}(Q^1)^\varrho & \le \sum_{j=0}^{M}\sum_{i=j}^{M} L^{(e)}_{r,\infty}(P_{j,i})^\varrho \\ & \le c_1 \sum_{j=0}^{M} 2^{-j\varrho\delta} \sum_{i=j}^{M} 2^{-is'\varrho}\, 2^{in\varrho(\frac{1}{r}-\frac{1}{p})} \\ & \le c_2 \sum_{j=0}^{M} 2^{-j\varrho\delta}\, 2^{-Ms'\varrho}\, 2^{Mn\varrho(\frac{1}{r}-\frac{1}{p})} \\ & \le c_3\, 2^{-Ms'\varrho}\, 2^{Mn\varrho(\frac{1}{r}-\frac{1}{p})} \end{aligned}$$

if r is chosen such that

$$n\Big(\frac{1}{r}-\frac{1}{p}\Big) > s'$$

and this implies

$$e_{2^{Mn}}(Q^1) \le c_4\, 2^{nM(-\frac{1}{p_1}+\frac{1}{p_2}-\frac{s'}{n})}\,.$$

To estimate $L^{(e)}_{r,\infty}(Q^2)$, $L^{(e)}_{r,\infty}(P^2)$ and $L^{(e)}_{r,\infty}(Q^3)$ we assume except for special cases

$$n\Big(\frac{1}{r}-\frac{1}{p}\Big) < s' \qquad \text{and} \qquad d\Big(\frac{1}{r}-\frac{1}{p}\Big) < \delta\,. \qquad (*)$$

Because of

$$s' > n/p^* = n\,\max(0,-1/p) \qquad \text{and} \qquad \delta > d/p^* = d\,\max(0,-1/p)$$

we have

$$\max\Big(\frac{n}{p},0\Big) < s' + \frac{n}{p} \qquad \text{and} \qquad \max\Big(\frac{d}{p},0\Big) < \delta + \frac{d}{p}\,.$$

Hence, there exists for $0 \le d \le n$ an appropriate r with

$$\max\Big(0,\frac{1}{p}\Big) < \frac{1}{r}$$

and

$$n\Big(\frac{1}{r}-\frac{1}{p}\Big) < s' \qquad \text{and} \qquad d\Big(\frac{1}{r}-\frac{1}{p}\Big) < \delta\,.$$

Estimate of $L^{(e)}_{r,\infty}(Q^2)$. We obtain for $0 \le d \le n$ and with $(*)$

$$
\begin{aligned}
L^{(e)}_{r,\infty}(Q^2)^{\varrho} &\le c_2 \sum_{j=0}^{M} 2^{-j\varrho\delta} \sum_{i=M+1}^{\infty} 2^{-is'\varrho}\, 2^{in\varrho(\frac{1}{r}-\frac{1}{p})} \\
&\le c_3 \sum_{j=0}^{M} 2^{-j\varrho\delta}\, 2^{-(M+1)s'\varrho}\, 2^{(M+1)n\varrho(\frac{1}{r}-\frac{1}{p})} \\
&\le c_4\, 2^{-Ms'\varrho}\, 2^{Mn\varrho(\frac{1}{r}-\frac{1}{p})}
\end{aligned}
$$

This gives

$$
e_{2^{Mn}}(Q^2) \le c_4\, 2^{nM(-\frac{1}{p_1}+\frac{1}{p_2}-\frac{s'}{n})}\,.
$$

Estimate of $L^{(e)}_{r,\infty}(P^2)$. We assume for $0 < d \le n$ condition $(*)$

$$\begin{aligned} L^{(e)}_{r,\infty}(P^2)^\varrho &\le c_2 \sum_{j=M+1}^{\infty} 2^{-j\varrho\delta} 2^{\varrho dj(\frac{1}{r}-\frac{1}{p})} \sum_{i=0}^{j-1} 2^{-is'\varrho}\, 2^{i(n-d)\varrho(\frac{1}{r}-\frac{1}{p})} \\ &\le c_3 \sum_{j=M+1}^{\infty} 2^{-j\varrho\delta} 2^{\varrho dj(\frac{1}{r}-\frac{1}{p})} \\ &\le c_4\, 2^{-M\delta\varrho}\, 2^{Md\varrho(\frac{1}{r}-\frac{1}{p})} \end{aligned}$$

This gives

$$e_{2^{Md}}(P^2) \le c_4\, 2^{dM(-\frac{1}{p_1}+\frac{1}{p_2}-\frac{\delta}{d})}\,. \qquad d \ne 0$$

In case $d = 0$ we chose instead of condition $(*)$ the parameter r such that

$$s' + \delta > n\Big(\frac{1}{r} - \frac{1}{p}\Big) > s'$$

holds and obtain in this case

$$e_{2^{Mn}}(P^2) \le c_4\, 2^{nM(-\frac{1}{p_1}+\frac{1}{p_2}-\frac{s'}{n}-\frac{\delta}{n})}\,.$$

Estimate of $L^{(e)}_{r,\infty}(Q^3)$. We assume for $0 \le d \le n$ again $(*)$

$$\begin{aligned} L^{(e)}_{r,\infty}(Q^3)^{\varrho} &\le c_2 \sum_{j=M+1}^{\infty} 2^{-j\varrho\delta} \sum_{i=j}^{\infty} 2^{-is'\varrho}\, 2^{in\varrho(\frac{1}{r}-\frac{1}{p})} \\ &\le c_3 \sum_{j=M+1}^{\infty} 2^{-j\varrho\delta}\, 2^{-js'\varrho}\, 2^{jn\varrho(\frac{1}{r}-\frac{1}{p})} \\ &\le c_4\, 2^{-(M+1)\varrho\delta}\, 2^{-(M+1)s'\varrho}\, 2^{(M+1)n\varrho(\frac{1}{r}-\frac{1}{p})} \end{aligned}$$

This gives

$$e_{2^{Mn}}(Q^3) \le c_4\, 2^{nM(-\frac{1}{p_1}+\frac{1}{p_2}-\frac{\delta}{n}-\frac{s'}{n})}\,.$$

For the estimate from below we consider again a commutative diagram

$$\begin{array}{ccc} \ell_{p_1}^{M_{j,i}} & \xrightarrow{S_{j,i}} & \ell_{q_1}(2^{j\delta}\ell_{p_1}(w)) \\ id_1 \downarrow & & \downarrow id \\ \ell_{p_2}^{M_{j,i}} & \xleftarrow{T_{j,i}} & \ell_{q_2}(\ell_{p_2}) \end{array}$$

Observe

$$\left\| \, T_{j,i} \left| \ell_{q_2}(\ell_{p_2}) \to \ell_{p_2}^{M_{j,i}} \right\| \right. \quad = \quad 1$$

and

$$\left\| \, S_{j,i} \left| \ell_{p_1}^{M_{j,i}} \to \ell_{q_1}(2^{j\delta}\ell_{p_1}(w)) \right\| \right. \quad \sim \quad 2^{j\delta}2^{is'} \, .$$

Using once more the characterization of $e_k\big(\mathrm{id}\,_1 : \ell_{p_1}^N \to \ell_{p_2}^N\big)$ we find with $k = 2^{in+(j-i)d} \sim M_{j,i}$ in case $i = 0$

$$k^{-\frac{\delta}{d}-\frac{1}{p_1}+\frac{1}{p_2}} \ \leq\ c\, e_k\big(\mathrm{id}\ :\ \ell_{q_1}\big(2^{j\delta}\,\ell_{p_1}(w)\big) \hookrightarrow \ell_{q_2}(\ell_{p_2})\big) \qquad \text{if} \quad d \neq 0$$

and $k = 2^{in} \sim M_{j,i}$ in case $j = 0$

$$k^{-\frac{s'}{n}-\frac{1}{p_1}+\frac{1}{p_2}} \ \leq\ c\, e_k\big(\mathrm{id}\ :\ \ell_{q_1}\big(2^{j\delta}\,\ell_{p_1}(w)\big) \hookrightarrow \ell_{q_2}(\ell_{p_2})\big) \qquad .$$

Consequently

$$e_k(id : \ell_{q_1}\Big(2^{j\delta}\,\ell_{p_1}(w)\Big) \hookrightarrow \ell_{q_2}(\ell_{p_2}))\ \sim\ k^{-\frac{1}{p_1}+\frac{1}{p_2}-\min\left(\frac{\delta}{d},\frac{s'}{n}\right)}$$

Theorem: Let U be a d-set , $0 \leq d \leq n$, $\frac{1}{p^*} := \left(\frac{1}{p_2} - \frac{1}{p_1}\right)_+$,
$\delta = s_1 - s_2 - n(1/p_1 - 1/p_2) > d/p*$, $s' = s'_1 - s'_2 > n/p*$
and $w_{\ell,j}(x) = (1 + 2^j \text{dist}(x, U))^{s'_\ell}$ $\ell = 1\,, 2\,.$
Then

$$e_k(id : B^{s_1,mloc}_{p_1,q_1}(\mathbb{R}^n, w^{s'_1}) \hookrightarrow B^{s_2,mloc}_{p_2,q_2}(\mathbb{R}^n, w^{s'_2})) \sim k^{-\frac{1}{p_1}+\frac{1}{p_2}-\min\left(\frac{\delta}{d},\frac{s'}{n}\right)} \quad .$$

Corollary: If $U = \{x_0\}$, then

$$e_k(id : B^{s_1,s'_1}_{p_1,q_1}(\mathbb{R}^n, \{x_0\}) \hookrightarrow B^{s_2,s'_2}_{p_2,q_2}(\mathbb{R}^n, \{x_0\})) \sim k^{-\frac{1}{p_1}+\frac{1}{p_2}-\frac{s'_1-s'_2}{n}}$$

and particularly, if $s_1 - s_2 > n/2$ and $s'_1 > s'_2$, then

$$e_k(id : H^{s_1,s'_1}_{x_0}(\mathbb{R}^n) \hookrightarrow C^{s_2,s'_2}_{x_0}(\mathbb{R}^n)) \sim k^{-\frac{1}{2}-\frac{s'_1-s'_2}{n}} \quad .$$

THANK YOU FOR YOUR ATTENTION