Lusin’s theorem and compactness criteria in spaces of measurable functions

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The set $S \subset X$ in complete metric space $X$ is

— *compact* if any open cover of $S$ contains the finite subcover,

— *completely bounded* if for any $\varepsilon > 0$ there exists finite $\varepsilon$-net of $S$.

Compact set is bounded and closed.

The following statements are equivalent

1) $S$ is compact,

2) $S$ is closed and completely bounded (Hausdorff criterion),

3) $S$ is closed and any infinite subset have a limit point (Bolzano–Weierstrass property).

We will use the terms of completely boundedness.
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We will use the terms of completely boundedness.
Compactness in $C(X)$

Theorem (C. Arzela–G. Ascoli criterion)

Let $X$ be compact metric space. The set $S \subset C(X)$ is completely bounded if and only if

$$\exists M > 0 \quad \forall f \in S, \ x \in X \quad |f(x)| \leq M$$

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall f \in S$$

$$\forall x_1, x_2 \in X \quad d(x_1, x_2) < \delta \quad |f(x_1) - f(x_2)| < \varepsilon$$

Ascoli (1883) proved the sufficiency, Arzela (1889) proved the necessity of this condition.

The extension to a case of metric space was done by M. Fréchet (1906).
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Notations

Let \((X, d, \mu)\) be bounded metric space (\(\text{diam } X = 1\)) with metric \(d\) and regular Borel measure \(\mu\),

\[
B = B(x, r) = \{y \in X : d(x, y) < r\},
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\(r_B\) is the radius of \(B\),

\[
f_B = \int_B f \, d\mu = \frac{1}{\mu(B)} \int_B f \, d\mu
\]

Doubling condition

\[
\mu(B(x, 2r)) \leq c_\mu \mu(B(x, r)), \quad x \in X, \quad r > 0.
\] (1)

\[
\|f\|_{L^p} = \|f\|_p = \left( \int_X |f|^p \, d\mu \right)^{1/p}, \quad p > 0.
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Let $X \subset \mathbb{R}^n$ be bounded and measurable. All functions are zero outside of $X$.

**Theorem (M. Riesz criterion)**

$S \subset L^p(X)$, $p > 0$, is completely bounded if and only if $S$ is bounded and

\[
\lim_{|h| \to 0} \sup_{f \in S} \int_X |f(x + h) - f(x)|^p \, dx = 0.
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M. Riesz (1933) $p \geq 1$, M. Tsuji (1951) $0 < p < 1$. 
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M.Riesz theorem and compactness

Maximal functions and compactness

The space of measurable functions

Final remarks
A.N. Kolmogorov criterion

Let $X \subset \mathbb{R}^n$ be bounded and measurable. All functions are zero outside of $X$.

**Theorem (A.N. Kolmogorov criterion)**

$S \subset L^p(X)$, $p \geq 1$, is completely bounded if and only if $S$ is bounded and

$$\lim_{r \to +0} \sup_{f \in S} \int_{X} \left| f(x) - \int_{B(x,r)} f d\mu \right|^p d\mu(x) = 0.$$

A.N. Kolmogorov (1931),
A. Kałamajska (1999), $\forall r > 0 \inf_{x \in X} \mu B(x, r) > 0$ (only sufficiency).
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Local smoothness inequality

Ω be the class of functions \( \eta : (0, 1] \to \mathbb{R}_+, \eta(+0) = 0, \)

\[ \eta(r) \uparrow, \quad \exists a > 0 \quad \eta(r)r^{-a} \downarrow. \]

Such function we call ”smoothness function”.

If \( f \) is measurable function on \( X \), then denote by \( D_\eta(f) \) the set of all measurable functions \( g \geq 0 \) such that

\[ \exists E \subset X \quad \mu E = 0 \]

\[ |f(x) - f(y)| \leq [g(x) + g(y)]\eta(d(x,y)), \quad x, y \in X \setminus E. \quad (2) \]

(2) is called local smoothness inequality.
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Quantitative form of Luzin theorem

The following statement is the quantitative form of Luzin theorem: for any measurable function $f$ there exist $\eta \in \Omega$ such that $D_\eta(f) \neq \emptyset$.

Denote by

$$E_\lambda = \{x \in X \setminus E : g(x) \leq \lambda\},$$

then $\mu E_\lambda \to \mu X$ ($\lambda \to \infty$) and it follows from (2)

$$|f(x) - f(y)| \leq 2\lambda \eta(d(x,y)), \quad x, y \in E_\lambda.$$

Function $f$ is uniformly continuous on $E_\lambda$ and $2\lambda \eta(\delta)$ is the upper estimate for modulus of continuity of $f$ on this set.
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New criteria in terms of Luzin theorem

**Theorem**

The set $S \subset L^p(X)$, $p > 0$, is completely bounded if and only if $S$ is bounded and

$$\exists \eta \in \Omega \quad \sup_{f \in S} \inf_{g \in D_\eta(f)} \|g\|_{L^p(X)} < +\infty$$

If we replace here $L^p$ by $C$, then we obtain perfectly Arzela–Ascoli criterion for $C(X)$!
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Maximal operators

For $q > 0$ and $\eta \in \Omega$ denote by

$$\mathcal{N}_q^\eta f(x) = \sup_{B \ni x} \frac{1}{\eta(rB)} \left( \int_B |f(x) - f(y)|^q d\mu(y) \right)^{1/q}.$$

$X = \mathbb{R}^n$, $\eta(t) = t^\alpha$ A.Cálderon (1972), A.Cálderon–R.Scott (1978),
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$$|f(x) - f(y)| \leq c_q [\mathcal{N}_q^\eta f(x) + \mathcal{N}_q^\eta f(x)] \eta(d(x, y)).$$
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Criterion in terms of maximal operators

**Theorem**

Let $0 < q < p$. The set $S \subset L^p(X)$, $p > 0$, is completely bounded if and only if $S$ is bounded and

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Kolmogorov criterion is false if the set $X \subset \mathbb{R}^n$ is unbounded. J. Tamarkin (1932) shows that here we need extra condition for compactness

$$\lim_{R \to +\infty} \sup_{f \in S} \int_{|x| > R} |f|^p \, d\mu = 0.$$ 

A. Tulajkov (1933) proved this results of Kolmogorov–Tamarkin for $p = 1$. Similarly in all above criteria for $L^p(X)$ we need additional condition

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for some $x_0 \in X$. 

V.G.Krotov  Oppurg, October 10-16, 2010
Case of unbounded $X$

Kolmogorov criterion is false if the set $X \subset \mathbb{R}^n$ is unbounded. J.Tamarkin (1932) shows that here we need extra condition for compactness

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\lim_{R \to +\infty} \sup_{f \in S} \int_{|x| > R} |f|^p \, d\mu = 0.
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for some $x_0 \in X$. 
The space $L^0$

$L^0(X)$ be the set of all (equivalence classes) measurable functions $f : X \to \mathbb{R}$.

$L^0(X)$ is complete with respect to the metric

$$d_{L^0}(f, g) = \int_X \frac{|f - g|}{1 + |f - g|} d\mu.$$ 

Convergence in $L^0(X)$ coincide with convergence on measure

$$\forall \varepsilon > 0 \lim_{n \to \infty} \mu \{|f - f_n| > \varepsilon\} = 0.$$ 

Convergence on measure was introduced by F.Riesz (1909).

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A. Lebesgue–G. Vitali criterion

Let $X$ be any set with finite measure.

Theorem (A. Lebesgue–G. Vitali)

The set $S \subset L^p(X)$, $p > 0$, is completely bounded if and only if $S$ is completely bounded in $L^0(X)$ and

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\lim_{\mu E \to 0} \sup_{f \in S} \int_E |f|^p \, d\mu = 0
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(property of equiabsolutely continuity).
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Fréchet criterion

**Theorem (M.Fréchet)**

\( S \subseteq L^0(X) \) is completely bounded if and only if it is almost uniformly bounded and almost equicontinuous, that is

\[
\forall \varepsilon > 0 \quad \exists \delta > 0, \lambda > 0 \quad \forall f \in S \quad \exists E(f) \subseteq X
\]

\( a) \mu E(f) < \varepsilon, \)

\( b) |f(x) - f(y)| < \varepsilon \text{ for } x, y \in X \setminus E(f), d(x, y) < \delta, \)

\( c) |f(x)| \leq \lambda \text{ for } x \in X \setminus E(f). \)

M.Fréchet (1927), E.Hanson (1933) give simple proof.
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Theorem

\[ S \subset L^0(X) \text{ is completely bounded if and only if } \]

\[ \lim_{\lambda \to +\infty} \sup_{f \in S} \mu\{|f| > \lambda\} = 0 \]

and

\[ \exists \eta \in \Omega \quad \lim_{\lambda \to +\infty} \sup_{f \in S} \inf_{g \in D_\eta(f)} \mu\{g > \lambda\} = 0. \]

This is quantitative form of Fréchet criterion.
Criterion with local smoothness

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This is quantitative form of Fréchet criterion.
Classes $\varphi(L)$

Let $\Phi$ be the set of all functions $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$

$$\varphi(t) \uparrow, \quad \varphi(0) = \varphi(+0) = 0, \quad \lim_{t \to +\infty} \varphi(t) = +\infty.$$ 

$$\Phi_1 = \{ \varphi \in \Phi : \varphi(t)/t \downarrow \}.$$ 

$$\varphi(L) = \left\{ f \in L^0(X) : \int_X \varphi(|f|)d\mu < +\infty \right\}.$$ 

If $\varphi \in \Phi_1$ then $\varphi(L)$ is complete metric space with respect to metric

$$d_\varphi(f, g) = \int_X \varphi(|f - g|)d\mu.$$
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Criterion with \( \varphi(L) \)

**Theorem**

The set \( S \subset L^0(X) \) is completely bounded if and only if there exists function \( \varphi \in \Phi_1 \) such that \( S \) is completely bounded in \( \varphi(L) \).
Criterion with maximal functions

\[ N^\varphi_\eta f(x) = \sup_{B \ni x} \frac{1}{\eta(rB)} \int_B \varphi(f(x) - f(y)) \, d\mu(y). \]

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The set \( S \subset L^0(X) \) is completely bounded if and only if

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**Theorem**

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Classes $C^p_\eta$

For $0 < q < p$ denote by

$$C^p_{\eta,q} = \{ f \in L^0(X) : f, \mathcal{N}_\eta^q f \in L^p(X) \}.$$

$S \subset L^p$ completely bounded

$\quad \quad \uparrow$

$\exists \eta \in \Omega \quad S$ is bounded in $C^p_{\eta,q}$
Classes \( C_{\eta}^{p,q} \)

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\]

\( S \subset L^p \quad \text{completely bounded} \)

\[
\uparrow
\]

\[ \exists \eta \in \Omega \quad S \text{ is bounded in } C_{\eta}^{p,q} \]
Under some natural restriction on $\eta \in \Omega$ it is easy to see that

$$|f(x) - T_n f(x)| \leq c\eta \left( \frac{1}{n} \right) \mathcal{N}^1_\eta f(x), \quad x \in X.$$ 

Here $T_n f$ may be for example Jackson polynomials, or Cesaro means of Fourier series of function $f$ and so on.

$$\|f - T_n f\|_{L^p} \leq c\eta \left( \frac{1}{n} \right) \|\mathcal{N}^1_\eta f\|_{L^p}.$$
Pointwise approximation

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