Lusin's theorem and compactness criteria in spaces of measurable functions

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Compactness

The set $S \subset X$ in complete metric space X is
— *compact* if any open cover of S contains the finite subcover,
— *completely bounded* if for any $\varepsilon > 0$ there exists finite ε-net of S.
Compact set is bounded and closed.
The following statements are equivalent
1) S is compact,
2) S is closed and completely bounded (Hausdorff criterion),
3) S is closed and any infinite subset have a limit point (Bolzano–Weierstrass property).
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Compactness in $C(X)$

Theorem (C.Arzela–G.Ascoli criterion)

Let X be compact metric space. The set $S \subset C(X)$ is completely bounded if and only if

$$\exists\, M > 0 \quad \forall\, f \in S,\ x \in X \quad |f(x)| \le M$$

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, f \in S$$

$$\forall\, x_1, x_2 \in X\ d(x_1, x_2) < \delta \quad |f(x_1) - f(x_2)| < \varepsilon$$

Ascoli (1883) proved the sufficiency,
Arzela (1889) proved the necessity of this condition.
The extension to a case of metric space was done by M.Fréchet (1906).

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Notations

Let (X, d, μ) be bounded metric space $(\operatorname{diam} X = 1)$ with metric d and regular Borel measure μ,

$$B = B(x, r) = \{y \in X : d(x, y) < r\},$$

r_B is the radius of B,

$$f_B = \fint_B f\,d\mu = \frac{1}{\mu B} \int_B f\,d\mu$$

Doubling condition

$$\mu B(x, 2r) \leq c_\mu \mu B(x, r), \quad x \in X, \quad r > 0. \tag{1}$$

$$\|f\|_{L^p} = \|f\|_p = \left(\int_X |f|^p d\mu \right)^{1/p}, \quad p > 0.$$

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Let $X \subset \mathbb{R}^n$ be bounded and measurable. All functions are zero outside of X.

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$S \subset L^p(X)$, $p > 0$, is completely bounded if and only if S is bounded and

$$\lim_{|h| \to +0} \sup_{f \in S} \int_X |f(x+h) - f(x)|^p dx = 0.$$

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Local smoothness inequality

Ω be the class of functions $\eta : (0,1] \to \mathbb{R}_+$, $\eta(+0) = 0$,

$$\eta(r) \uparrow, \quad \exists\, a > 0 \quad \eta(r) r^{-a} \downarrow .$$

Such function we call "smoothness function".

If f is measurable function on X, then denote by $D_\eta(f)$ the set of all measurable functions $g \geq 0$ such that

$$\exists\, E \subset X \quad \mu E = 0$$

$$|f(x) - f(y)| \leq [g(x) + g(y)]\eta(d(x,y)), \quad x, y \in X \setminus E. \tag{2}$$

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Quantitative form of Luzin theorem

The following statement is the quantitative form of Luzin theorem: **for any measurable function f there exist $\eta \in \Omega$ such that $D_\eta(f) \neq \varnothing$.**

Denote by

$$E_\lambda = \{x \in X \setminus E : g(x) \le \lambda\},$$

then $\mu E_\lambda \to \mu X$ $(\lambda \to \infty)$ and it follows from (2)

$$|f(x) - f(y)| \le 2\lambda\eta(d(x,y)), \quad x, y \in E_\lambda.$$

Function f is uniformly continuous on E_λ and $2\lambda\eta(\delta)$ is the upper estimate for modulus of continuity of f on this set.

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New criteria in terms of Luzin theorem

Theorem

The set $S \subset L^p(X)$, $p > 0$, *is completely bounded if and only if* S *is bounded and*

$$\exists\, \eta \in \Omega \quad \sup_{f \in S} \inf_{g \in D_\eta(f)} \|g\|_{L^p(X)} < +\infty$$

If we replace here L^p by C, then we obtain perfectly Arzela–Ascoli criterion for $C(X)$!

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Maximal operators

For $q > 0$ and $\eta \in \Omega$ denote by

$$\mathcal{N}_\eta^q f(x) = \sup_{B \ni x} \frac{1}{\eta(r_B)} \left(\fint_B |f(x) - f(y)|^q d\mu(y) \right)^{1/q}.$$

$X = \mathbb{R}^n$, $\eta(t) = t^\alpha$ A.Cálderon (1972), A.Cálderon–R.Scott (1978),
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$$|f(x) - f(y)| \le c_q[\mathcal{N}_\eta^q f(x) + \mathcal{N}_\eta^q f(x)]\eta(d(x,y)).$$

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Kolmogorov criterion is false if the set $X \subset \mathbb{R}^n$ is unbounded.

J.Tamarkin (1932) shows that here we need extra condition for compactness

$$\lim_{R\to+\infty} \sup_{f\in S} \int_{|x|>R} |f|^p \, d\mu = 0.$$

A.Tulajkov (1933) proved this results of Kolmogorov–Tamarkin for $p = 1$. Similarly in all above criteria for $L^p(X)$ we need additional condition

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$$\lim_{R\to+\infty} \sup_{f\in S} \int_{X\setminus B(x_0,R)} |f|^p \, d\mu = 0,$$

for some $x_0 \in X$.

The space L^0

$L^0(X)$ be the set of all (equivalence classes) measurable functions $f : X \to \mathbb{R}$.
$L^0(X)$ is complete with respect to the metric

$$d_{L^0}(f, g) = \int_X \frac{|f-g|}{1+|f-g|} d\mu.$$

Convergence in $L^0(X)$ coincide with convergence on measure

$$\forall \varepsilon > 0 \quad \lim_{n\to\infty} \mu\{|f - f_n| > \varepsilon\} = 0.$$

Convergence on measure was introduced by F.Riesz (1909).
The space $L^0(X)$ was defined by M.Fréchet (1919).

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A.Lebesgue–G.Vitali criterion

Let X be any set with finite measure.

Theorem (A.Lebesgue–G.Vitali)

The set $S \subset L^p(X)$, $p > 0$, is completely bounded if and only if S is completely bounded in $L^0(X)$ and

$$\lim_{\mu E \to 0} \sup_{f \in S} \int_E |f|^p d\mu = 0$$

(property of equiabsolutely continuity).

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Fréchet criterion

Theorem (M.Fréchet)

$S \subset L^0(X)$ is completely bounded if and only if it is almost uniformly bounded and almost equicontinuous, that is

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0,\ \lambda > 0 \quad \forall\, f \in S \quad \exists\, E(f) \subset X$$

a) $\mu E(f) < \varepsilon$,
b) $|f(x) - f(y)| < \varepsilon$ *for* $x, y \in X \setminus E(f)$, $d(x, y) < \delta$,
c) $|f(x)| \le \lambda$ *for* $x \in X \setminus E(f)$.

M.Fréchet (1927), E.Hanson (1933) give simple proof.

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Criterion with local smoothness

Theorem

$S \subset L^0(X)$ is completely bounded if and only if

$$\lim_{\lambda \to +\infty} \sup_{f \in S} \mu\{|f| > \lambda\} = 0$$

and

$$\exists\, \eta \in \Omega \qquad \lim_{\lambda \to +\infty} \sup_{f \in S} \inf_{g \in D_\eta(f)} \mu\{g > \lambda\} = 0.$$

This is quantitative form of Freće criterion.

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This is quantitative form of Freće criterion.

Classes $\varphi(L)$

Let Φ be the set of all functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$

$$\varphi(t) \uparrow, \quad \varphi(0) = \varphi(+0) = 0, \quad \lim_{t \to +\infty} \varphi(t) = +\infty.$$

$$\Phi_1 = \{\varphi \in \Phi : \varphi(t)/t \downarrow\}.$$

$$\varphi(L) = \left\{ f \in L^0(X) : \int_X \varphi(|f|) d\mu < +\infty \right\}.$$

If $\varphi \in \Phi_1$ then $\varphi(L)$ is complete metric space with respect to metric

$$d_\varphi(f, g) = \int_X \varphi(|f - g|) d\mu.$$

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The set $S \subset L^0(X)$ is completely bounded if and only if there exists function $\varphi \in \Phi_1$ such that S is completely bounded in $\varphi(L)$.

Criterion with maximal functions

$$\mathcal{N}_\eta^\varphi f(x) = \sup_{B \ni x} \frac{1}{\eta(r_B)} \int_B \varphi(f(x) - f(y))\, d\mu(y).$$

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For $0 < q < p$ denote by

$$C_\eta^{p,q} = \left\{ f \in L^0(X) : f, \mathcal{N}_\eta^q f \in L^p(X) \right\}.$$

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Pointwise approximation

Under some natural restriction on $\eta \in \Omega$ it is easy to see that

$$|f(x) - T_n f(x)| \leq c\eta\left(\frac{1}{n}\right)\mathcal{N}_\eta^1 f(x), \quad x \in X.$$

Here $T_n f$ may be for example Jackson polynomials, or Cesaro means of Fourier series of function f and so on.

$$\|f - T_n f\|_{L^p} \leq c\eta\left(\frac{1}{n}\right)\left\|\mathcal{N}_\eta^1 f\right\|_{L^p}.$$

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