Faber type functions on distinguished fractals

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Smoothness, Approximation and Function Spaces
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1. Besov spaces on d-sets
2. Self-similar sets
   - Definition
   - Dirichlet forms
   - Harmonic functions
3. Characterization of Besov spaces

Definition

A compact set $\Gamma \subset \mathbb{R}^n$ is called a d-set, $0 < d < n$, if there is a Radon measure μ in $\mathbb{R}^n$ with

$$\operatorname{supp} \mu = \Gamma \;\text{ and }\; \mu(B(\gamma, r)) \sim r^d, \;\; \gamma \in \Gamma, \;\; 0 < r \leq 1, \qquad (1)$$

where $B(\gamma, r)$ is a ball in $\mathbb{R}^n$ centered at γ and of radius r.

The Hausdorff dimension of Γ equals d,

$$\dim_{\mathrm{H}} \Gamma = d,$$

and the restriction H^d on Γ satisfies (1).

Let $k \in \mathbb{N}$

$$0 \le k < s \le k+1, \;\; 1 \le p, q \le \infty.$$

A function $f \in B^s_{pq}(\mathbb{R}^n)$, if

$$\|f|B^s_{pq}(\mathbb{R}^n)\| = \sum_{|\alpha|\le k} \|D^\alpha f|L_p(\mathbb{R}^n)\| + \sum_{|j|=k} \left(\int_{\mathbb{R}^n} \frac{\|\Delta_h^2 D^j f|L_p(\mathbb{R}^n)\|^q}{|h|^{n+(s-k)q}}\, dh \right)^{1/q}$$

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$$S(\mathbb{R}^n) = \{ f \in C^\infty(\mathbb{R}^n) : p_N(f) < \infty \}$$

$$p_N(f) = \sup_{|\alpha| \le N} \sup_{x \in \mathbb{R}^n} \left(1 + |x|^2\right)^N |D^\alpha f(x)|$$

$S(\mathbb{R}^n)$ is dense in $B^s_{pq}(\mathbb{R}^n)$.

Definition

Let Γ be a d-set and

$$s > 0, \quad 1 < p < \infty, \quad 0 < q < \infty. \tag{2}$$

Let for some $c > 0$,

$$\int_\Gamma |\varphi(\gamma)| \; \mu(d\gamma) \le c\|\varphi|B^s_{pq}(\mathbb{R}^n)\|, \;\; \text{for all } \varphi \in S(\mathbb{R}^n).$$

Then the trace operator

$$\mathrm{tr}_\mu : B^s_{pq}(\mathbb{R}^n) \hookrightarrow L_1(\Gamma, \mu),$$

is the completion of the pointwise trace $(\mathrm{tr}_\mu \, \varphi)(\gamma) = \varphi(\gamma)$ with $\varphi \in S(\mathbb{R}^n)$. $g \in \mathrm{tr}_\mu \, B^s_{pq}(\mathbb{R}^n) \subset L_1(\Gamma, \mu)$ is quasi-normed by

$$\|g|\,\mathrm{tr}_\mu \, B^s_{pq}(\mathbb{R}^n)\| = \inf \left\{ \|f|B^s_{pq}(\mathbb{R}^n)\| : \mathrm{tr}_\mu \, f = g \right\}.$$

Definition

Let $1 < p < \infty$, $0 < q < \infty$ and $s > 0$. Then

$$B^s_{pq}(\Gamma, \mu) = \mathrm{tr}_\mu\, B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$$

and

$$B^0_{p1}(\Gamma, \mu) = \mathrm{tr}_\mu\, B^{\frac{n-d}{p}}_{p1}(\mathbb{R}^n) = L_p(\Gamma, \mu).$$

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$B^s_{pp}(\Gamma, \mu)$ with $1 < p < \infty$ and $0 < s < 1$ can be equivalently normed by

$$\|f|B^s_{pp}(\Gamma, \mu)\|^p_* = \int_\Gamma |f(\gamma)|^p \, \mu(d\gamma) + \int_\Gamma \int_\Gamma \frac{|f(\gamma) - f(\delta)|^p}{|\gamma - \delta|^{d+sp}} \, \mu(d\gamma)\mu(d\delta)$$

with $\mu = \mathrm{H}^d|_\Gamma$.

Definition

$F : \mathbb{R}^n \to \mathbb{R}^n$ is called a similarity (similitude), if there is a constant $0 < \rho < 1$ such that for all $x, y \in \mathbb{R}^n$ holds

$$|F(x) - F(y)| = \rho\,|x - y|\,.$$

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Theorem

Let $\{F_i\}_{i=1}^N$ be similarities in $\mathbb{R}^n$. Then there exists a unique non-empty compact set $\Gamma \subset \mathbb{R}^n$ that satisfies

$$\Gamma = \bigcup_{i=1}^{N} F_i(\Gamma). \tag{3}$$

Γ is called a self-similar set with respect to $\{F_i\}_{i=1}^N$.

j-simplex:

$$\Gamma_w = F_w(\Gamma) = F_{w_1} \circ F_{w_2} \circ \ldots \circ F_{w_j}(\Gamma)$$

Shift space:

$$\Sigma = \{(\omega_1, \omega_2, \ldots) : \omega_i \in \{1, 2, \ldots, N\}$$

$\sigma : \Sigma \to \Sigma$

$$\sigma(\omega_1, \omega_2, \ldots) = (\omega_2, \omega_3, \ldots)$$

$\pi : \Sigma \to \Gamma$

$$\pi(\omega) = \bigcap_{m=1}^{\infty} \Gamma_{\omega_1 \omega_2 \ldots \omega_m}$$

Critical and post critical set:

$$C = \bigcup_{i \neq j} (\Gamma_i \cap \Gamma_j)$$

$$\mathcal{C} = \pi^{-1}(C), \quad \mathcal{P} = \bigcup_{n \geq 1} \sigma^n(\mathcal{C})$$

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The boundary of Γ:

$$V_0 = \pi(\mathcal{P}), \;\; V_j = \bigcup_{i=1}^{N} F_i(V_{j-1})$$

$$V_* = \bigcup_{j=0}^{\infty} V_j, \quad \Gamma = \overline{V_*}$$

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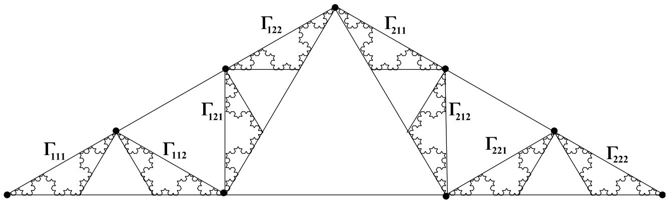

Suppose $u : V_j \to \mathbb{R}$, then there is a natural Dirichlet form

$$E_j(u) = \sum_{\xi \sim_j \eta} (u(\xi) - u(\eta))^2 .$$

Lemma

For every $u : V_j \to \mathbb{R}$ there exists a unique extension $\tilde{u} : V_{j+1} \to \mathbb{R}$ minimizing E_{j+1}:

$$E_{j+1}(\tilde{u}) = \min \left\{ E_{j+1}(u') : u'|_{G_j} = u \right\},$$

and

$$\alpha^j E_j(u) = \alpha^{j+1} E_{j+1}(\tilde{u}). \tag{4}$$

The renormalized graph energies are defined by

$$\mathcal{E}_j(u) = \alpha^j E_j(u).$$

Then (4) can be reformulated as

$$\mathcal{E}_j(u) = \mathcal{E}_{j+1}(\tilde{u}).$$

Definition

A continuous function $h : V_* \to \mathbb{R}$ is called harmonic if it minimizes $\mathcal{E}_j$ at all levels for given boundary values on V_0:

$$\mathcal{E}_j(h) = \min \{\mathcal{E}_j(u) : u|_{V_0} = \rho\}.$$

For any harmonic function u there exists a unique extension $\tilde{u} \in C(\Gamma)$ such that

$$\tilde{u}|_{V_*} = u|_{V_*}.$$

Definition

A continuous function $\psi : V_* \to \mathbb{R}$ is called piecewise harmonic of level j if $\psi \circ F_w$ is harmonic for all $|w| = j$.

Let ψ_ξ^j, $\xi \in V_j$, be a piecewise harmonic function of level j defined by

$$\psi_\xi^j(x) = \delta_{\xi x} = \begin{cases} 1, & x = \xi \\ 0, & x \in V_j \setminus \{\xi\}. \end{cases}$$

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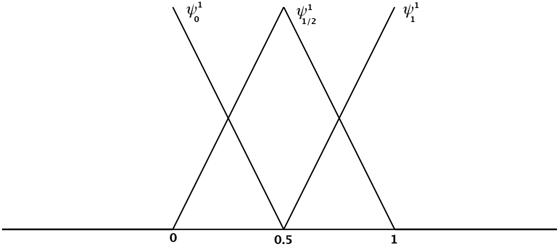

We additionally assume

$$\Gamma = \bigcup_{i=1}^{N} F_i(\Gamma), \quad \text{with } |F_i(x) - F_i(y)| = \rho\,|x - y|\,,$$

$$\operatorname{diam}\Gamma = 1.$$

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Properties of piecewise harmonic functions:

- $\operatorname{supp}\psi_\xi^j \subset B(\xi, \rho^j)$.
- $\left|\psi_\xi^j(x) - \psi_\xi^j(y)\right| \leq cR(x,y)$ for all $x, y \in \Gamma$, where $R(x,y)$ is the effective resistance metric. For certain fractals

$$R(x,y) \sim |x-y|^\sigma\,,$$

  thus

$$\left|\psi_\xi^j(x) - \psi_\xi^j(y)\right| \leq c\,|x-y|^\sigma\,.$$

Let $f \in C(\Gamma)$ and let $P_n f$, $n \geq 0$, be the unique piecewise harmonic function which interpolates f at all points in V_n:

$$P_0 f = \sum_{\xi \in V_0} f(x) \psi_\xi^0,$$

$$P_n f = \sum_{\xi \in V_0} f(x) \psi_\xi^0 + \sum_{j=1}^{n} \sum_{\xi \in V_j \setminus V_{j-1}} c_\xi(f) \psi_\xi^j, \quad n \geq 1,$$

with

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Set $V_{-1} = \emptyset$ and $P_{-1}f \equiv 0$, then

$$f = \sum_{j=0}^{\infty} \sum_{\xi \in V_j \setminus V_{j-1}} c_\xi(f)\psi_\xi^j.$$

Theorem

Let

$$1 < p < \infty \text{ and } \frac{d}{p} < s < \min\{1, \sigma\}, \tag{5}$$

Then $f \in C(\Gamma)$ *belongs to* $B_p^s(\Gamma)$ *if and only if it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{\xi \in V_j \setminus V_{j-1}} c_\xi(f) \psi_\xi^j,$$

where

$$C_p^s(f) = \left(\sum_{j=0}^{\infty} \rho^{j\left(\frac{d}{p} - s\right)p} \sum_{\xi \in V_j \setminus V_{j-1}} |c_\xi(f)|^p \right)^{\frac{1}{p}} < \infty$$

and unconditional convergence being in $C(\Gamma)$*. Furthermore,*

$$\|f | B_p^s(\Gamma)\| \sim C_p^s(f).$$

Thank you for your attention !!!