Embedding of Calderon-Kolyada classes C_η^p in $\varphi(L)$

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Outline

1. Introduction
   - Metric Measure Spaces
   - Smoothness parameters
   - Cálderon–Kolyada maximal functions
   - Classes $C^p_\eta(X)$
   - Classification of L^p-functions
2. Sobolev type embedding theorems
   - Embedding theorems
   - $C^p_\eta(X) \subset C^q_\sigma(X)$
   - $C^p_\eta(X) \subset L^q(X)$

Metric Measure Spaces

(X, d, μ) — bounded space of homogeneous type

$d : X \times X \to [0, \infty)$ — quasimetric

$$\exists\, a_d \geq 1 \;\; \forall\, x, y, z \in X$$

$$d(x, y) \leq a_d[d(x, z) + d(z, y)]$$

Doubling condition

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$$\mu\left(B\left(x,2r\right)\right) \leq c\mu\left(B\left(x,r\right)\right), \quad x \in X, \ \ r > 0.$$

$$\Downarrow$$

$$\exists \gamma > 0 \quad \forall x \in X \quad \forall 0 < r \leq R$$

$$\mu\left(B\left(x,R\right)\right) \leq c\left(\frac{R}{r}\right)^{\gamma} \mu\left(B\left(x,r\right)\right)$$

γ — "dimension" of X.

$$\operatorname{diam} X = 1, \quad \mu X = 1$$

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Smoothness parameters

$\Omega(a)$ — class of functions $\eta : (0, 2] \to \mathbb{R}_+$

$$\Omega(a) = \{\eta : \eta(+0) = 0, \quad \eta(t) \uparrow, \quad \eta(t)t^{-a} \downarrow\}$$

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Calderon–Kolyada maximal function

For $\eta \in \Omega$, $f \in L^1_{\mathrm{loc}}(X)$

$$\mathcal{N}_\eta f(x) = \sup_{B \ni x} \frac{1}{\eta(r_B)} \fint_B |f - f(x)| \, d\mu,$$

$X = \mathbb{R}^n$, $\eta(t) = t^\alpha$ — A.Cálderon (1972)

$X = [0,1]^n$, $\eta \in \Omega(1)$ — V.I.Kolyada (1987)

(X, d, μ), $\eta = t^\alpha$ — J.Hu and D.Yang (2003)
$\mathcal{N}_\eta$ measures local smoothness:

$$|f(x) - f(y)| \le \eta(d(x,y))[\mathcal{N}_\eta f(x) + \mathcal{N}_\eta f(y)]$$

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Classes $C^p_\eta(X)$

For $\eta \in \Omega$ and $p \geq 1$ define

$$C^p_\eta(X) = \{f \in L^p(X) : \|f\|_{C^p_\eta} = \|f\|_p + \|\mathcal{N}_\eta f\|_p < \infty\},$$

If $\eta(t) = t^\alpha$ we write $C^p_\alpha(X)$.
If $\eta(t) = t$, then

$$C^p_1(\mathbb{R}^n) = W^p_1(\mathbb{R}^n).$$

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Classification of L^p-functions

Theorem (I.A.Ivanishko, V.G.Krotov, 2007)

For $1 < p < \infty$

$$L^p(X) = \bigcup_{\eta \in \Omega} C^p_\eta(X).$$

Back to smoothness parameters

$X = [0,1]^n$, $\eta \in \Omega(1)$ — V.I.Kolyada (1987)

If $X = [0,1]^n$ or $X = \mathbb{R}^n$, $1 \le p < \infty$, $f \in C_\eta^p(X)$ and $\eta \in \Omega$ s.t.

$$\lim_{t \to +0} \frac{\eta(t)}{t} = 0,$$

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Two problems concerning $C^p_\eta(X)$

1. Relations between classes with different exponents?

2. If $\mathcal{N}_\eta f \in L^p(X)$, what kind of «good» properties of f can be expected?

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Relations between classes with different exponents

If $\eta \in \Omega$, $1 \leq p_1 \leq p_2 < \infty$, then

$$C_\eta^{p_2}(X) \subset C_\eta^{p_1}(X).$$

If $1 \leq p < \infty$, $\eta, \sigma \in \Omega$, s.t. $\eta(t) \leq c\sigma(t)$ then

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Embedding theorem

Recall classical Hardy-Littlewood-Sobolev embedding theorem

$$W_k^p(\mathbb{R}^n) \subset W_l^q(\mathbb{R}^n), \quad \frac{1}{q} = \frac{1}{p} - \frac{k-l}{n}.$$

A.Cálderon (1972)

$$C_\alpha^p(\mathbb{R}^n) \subset C_\beta^q(\mathbb{R}^n), \quad \eta(t) = t^\alpha, \quad \sigma(t) = t^\beta, \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha-\beta}{n}.$$

Theorem (I.A.Ivanishko, 2004)

Let $1 \le p < q < \infty$, $\sigma \in \Omega$ *and*

$$\eta(t) = \sigma(t) t^{\gamma\left(\frac{1}{p}-\frac{1}{q}\right)}. \tag{1}$$

Then $C_\eta^p(X) \subset C_\sigma^q(X)$.

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Embedding in L^q

Theorem (I.A.Ivanishko, 2004)

If $1 < p < q < \infty$ *and* $\eta(t) = t^{\gamma(\frac{1}{p}-\frac{1}{q})}$, *then*

$$C^p_\eta(X) \subset L^q(X).$$

C^p_η in $\varphi(L)$

$\varphi : [0, \infty) \to \mathbb{R}_+$ be increasing function, s.t. $\lim\limits_{t\to+\infty} \varphi(t) = +\infty$.

Define

$$\varphi(L) = \{f : X \to \mathbb{R} : \int\limits_X \varphi(|f|)d\mu < \infty\}.$$

Theorem (I.A.Ivanishko, V.G.Krotov 2010)

Let $1 < p < \infty$ *and* $\eta \in \Omega(\gamma/p)$, $\varphi : [0, \infty) \to \mathbb{R}_+$, *such that* $\varphi \uparrow$, $\lim\limits_{t\to+\infty} \varphi(t) = +\infty$ *and*

$$\varphi\left[\eta\left(\frac{1}{t^{p/\gamma}}\right)t\right] \le ct^p, \tag{2}$$

Then the following continious embedding holds

$$C^p_\eta(X) \subset \varphi(L).$$

The end

Thank you for your attention!