Tractability of Multivariate Integration via Importance Sampling

Workshop on Smoothness, Approximation, and Function Spaces

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The Problem

- $D \subseteq \mathbb{R}^d$ Borel measurable
- ϱ probability density on D
- H Hilbert space of functions $f : D \to \mathbb{R}$
- **Integration Problem:**

$$I(f) = \int_D f(x)\varrho(x)\,\mathrm{d}x$$

- Problem is well defined iff $H \subset L_1(\varrho)$ iff

$$C^{\mathrm{init}} = \left(\int_D \int_D K(x,y)\varrho(x)\varrho(y)\,\mathrm{d}x\mathrm{d}y \right)^{1/2} < \infty$$

- **Algorithms:** Randomized Algorithm using n function values, in particular importance sampling
- to have function values well defined we assume that H is a **reproducing kernel Hilbert space** with kernel $K : D \times D \to \mathbb{R}$.

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Importance Sampling

- another density function ω on D
- **Alternative Integration Problem:**

$$I(f) = \int_D \frac{f(x)\varrho(x)}{\omega(x)}\,\omega(x)\,\mathrm{d}x$$

- **Monte-Carlo:** $x_1, \ldots, x_n$ iid according to probability density ω

$$Q_n(f) = \frac{1}{n}\sum_{i=1}^{n} \frac{f(x_i)\varrho(x_i)}{\omega(x_i)}$$

- **Error:**

$$\begin{aligned} e_n^2 &= \sup_{\|f\|_H \le 1} \mathbb{E}\,|I(f) - Q_n(f)|^2 \\ &= \frac{1}{n} \sup_{\|f\|_H \le 1} \left(\int_D \frac{f(x)^2 \varrho(x)^2}{\omega(x)}\,\mathrm{d}x - I(f)^2 \right) \end{aligned}$$

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Analysis of the error

- Independent of the concrete integral $I(f)$:

$$e_n \leq n^{-1/2}\, C(\omega)$$

where $C(\omega)$ is given by

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- **Consequence:** Importance sampling has worst case error of order $n^{-1/2}$ if

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Known Result (Plaskota, Wasilkowski, Zhao 2009)

- use $|f(t)| \leq \sqrt{K(t,t)}$ for $\|f\| \leq 1$ to obtain

$$C(\omega) \leq \left(\int_D \frac{K(x,x)\varrho^2(x)}{\omega(x)} \, \mathrm{d}x \right)^{1/2}$$

- **Standard Monte-Carlo:** $\omega = \varrho$

$$C^{\mathrm{std}} := \left(\int_D K(x,x)\varrho(x) \, \mathrm{d}x \right)^{1/2} < \infty$$

sufficient for standard MC to have error of order $n^{-1/2}$

- **Optimization for ω:** with the condition

$$C^{\mathrm{sqrt}} = \int_D \sqrt{K(x,x)}\varrho(x) \, \mathrm{d}x < \infty$$

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The Result

- $C^{\text{init}} \leq C^{\text{imps}} \leq C^{\text{sqrt}} \leq C^{\text{std}}$
- $C^{\text{std}} < \infty \Longrightarrow$ Standard MC has error of order $n^{-1/2}$
- $C^{\text{sqrt}} < \infty \Longrightarrow$ PWZ-importance sampling has error of order $n^{-1/2}$
- $C^{\text{init}} < \infty \Longleftrightarrow H \subset L_1(\varrho) \Longleftrightarrow$ Problem is well defined $\Longleftrightarrow$ $J_H : H \to L_1(\varrho)$ is a bounded operator

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Theorem

If the initial error is finite or, equivalently, if the embedding $J_H : H \to L_1(\varrho)$ is a bounded operator then importance sampling has error of order $n^{-1/2}$. More precisely,

$$C^{\mathrm{imps}} \leq \sqrt{\tfrac{\pi}{2}}\, \|J_H : H \to L_1(\varrho)\|.$$

In particular, if the kernel K is nonnegative then

$$C^{\mathrm{imps}} \leq \sqrt{\tfrac{\pi}{2}}\, C^{\mathrm{init}}.$$

p-Summing Operators

- $1 \leq p < \infty$
- X, Y Banach spaces, $T : X \to Y$ bounded linear operator
- T is called p-summing if T maps weakly p-summable sequences in X to strongly p-summable sequences in Y.
- $$\sum_{i=1}^{n} \|Tx_i\|^p \leq c^p \sup_{\|a\|_{X'} \leq 1} \sum_{i=1}^{n} |a(x_i)|^p$$
- $\pi_p(T) = \inf c$
- **Pietsch Domination Theorem:** $T : X \to Y$ is p-summing if and only if there exists a constant $c \geq 0$ and a regular Borel probability measure ν on the weak-$*$-compact closed unit ball $B_{X'}$ of X' such that for all $x \in X$

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Change of Density

Theorem (Rosenthal, Johnson/Schechtman)

$X \subset L_1(\Omega,\mu)$, μ is a probability measure, $J : X \to L_1(\mu)$ bounded embedding. If the dual operator $J' : L_\infty(\Omega,\mu) \to X'$ is q-summing for some $1 \le q < \infty$ then there exists a measurable function $g > 0$ on Ω such that $\int_\Omega g \,\mathrm{d}\mu = 1$ and such that the isometry

$$M : L_1(\Omega,\mu) \to L_1(\Omega, g\,\mathrm{d}\mu) \ \text{ given by } Mf = fg^{-1}$$

maps X to a space $\widetilde{X} = M(X)$ which is contained in $L_p(\Omega, g\,\mathrm{d}\mu)$, where p is the dual index of q defined as $1/p + 1/q = 1$. Moreover, if we equip $\widetilde{X}$ with the norm from X, i.e. if we set

$$\|Mf|\widetilde{X}\| = \|f|X\| \ \text{ for } f \in X,$$

then the embedding $\widetilde{J} : \widetilde{X} \to L_p(\Omega, g\,\mathrm{d}\mu)$ has norm

$$\|\widetilde{J} : \widetilde{X} \to L_p(\Omega, g\,\mathrm{d}\mu)\| \le \pi_q(J' : L_\infty(\Omega,\mu) \to X').$$

The Method

- $C^{\text{init}} < \infty$ means $H \subset L_1(\varrho)$
- we want to change the density so that $H \subset L_2$
- recall: $J_H : H \to L_1(\varrho)$ is a linear bounded operator
- the **Little Grothendieck Theorem** tells you that the dual operator is 2-summing
- the **Change of Density Theorem** tells you that then the measure ϱdx can be changed with a density so that H then actually becomes a subspace of L_2
- that is exactly what we need
- the density ω for the importance sampling algorithm can be obtained from the **Pietsch measure** in the **Pietsch Domination Theorem** associated with the 2-summing operator J_H'

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Theorem

Assume that we have a sequence I_d of d-dimensional integration problems with normalized initial error. If the embeddings $J_{H_d} : H_d \to L_1(\varrho_d)$ are uniformly bounded, then the multivariate weighted integration problem is strongly polynomially tractable in the randomized setting with exponent 2. This is in particular the case if all the kernels K_d are nonnegative.

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Tractability of Multivariate Integration

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Toy Example

- $D = [0, \infty)$
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- $\varrho(x) = \sum_{j=1}^{\infty} r_j \, \mathbf{1}_j(x)$ for some $r_j \geq 0$ summing to 1
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Finally - A Real Example

- tractability of uniform integration on weighted Sobolev spaces
- $K_d(x,t) = \prod_{j=1}^d K^{\gamma_j}(x_j,t_j)$ for $x,t \in [0,1]^d$
- $K^\gamma(x,t) = 1 + \gamma \min(x,t)$ (non-periodic case) or $K^\gamma(x,t) = 1 + \gamma(\min(x,t) - xt)$ (periodic case) for $x,t \in [0,1]$
- $H_d = \otimes_{j=1}^d H^{\gamma_j}$ with H^γ consisting of absolutely continuous functions on $[0,1]$ whith first derivatives in $L_2[0,1]$ with norm

$$\|f\|_{H^\gamma}^2 = |f(0)|^2 + \frac{1}{\gamma}\int_0^1 |f'(x)|^2 \,\mathrm{d}x$$

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- Plaskota, Wasilkowski, Zhao 2009: importance sampling is strongly polynomial for the periodic case if $\sum \gamma_j^3 < \infty$
- Now: importance sampling is strongly polynomial without any condition in both the periodic and the non-periodic case

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Another Example - Smoothness sometimes does not help

- 1-dimensional space:

$$H_1 = \left\{ f : \mathbb{R} \to \mathbb{R} \,|\, f^{(k)}(0) = 0 \text{ for } k < r, f^{(r)} \in L_2(\mathbb{R}) \right\}$$

- **scalar product:**

$$\langle f, g \rangle_{H_1} = \int_{\mathbb{R}} f^{(r)}(x) g^{(r)}(x) \, \mathrm{d}x$$

- H_1 has a kernel K_1 which is **nonnegative** and **decomposable**:

$$K_1 \geq 0, \quad K_1(x, y) = 0 \text{ for } x \leq 0 \leq y$$

- d-dimensional space: H_d is the d-fold tensor product of H_1
- The problem is **Gaussian Integration**:

$$I_d(f) = \int_{\mathbb{R}^d} f(x) \prod_{j=1}^{d} \frac{\exp(-x_j^2/2)}{\sqrt{2\pi}} \, \mathrm{d}x$$

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- kernel is decomposable, so Novak/Woźniakowski show:

$$n^{\mathrm{ran}}(\varepsilon, I_d) \ge \frac{1}{8}\left(\frac{1}{\varepsilon}\right)^2 \text{ for } \varepsilon \in (0,1), d \ge \frac{2\log\varepsilon^{-1} - \log 2}{\log 2}$$

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The End

Thank you for your attention!