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SOME OPTIMAL EMBEDDINGS OF POTENTIALS

Abstract.

Spaces of potentials are considered in n-dimensional Euclidean space. They generalize the classical Riesz potentials [1]. Constructive criteria are obtained for the embeddings into rearrangement invariant spaces (RIS, see [2]), and optimal RIS are explicitly described for such embeddings. These results were announced in [3]. They are based on general approach from [4, 5] giving reduction of the problem to the properties of combined Hardy type operators on half axes $R_+ = (0, \infty)$ and on some sharp estimates for their norms on the cones of monotonic functions [6-11].

1. SPACE OF POTENTIALS

For $n \in N$ define class $\Im_n$ of continuous functions $\Phi: \; R_+ \to R_+$:

1) $0 \le \Phi \;\; \downarrow$;

2) there is $c_0 \in R_+$ such that

$$\int_0^r \Phi(\rho)\,\rho^{n-1}\,d\rho \le c_0\,\Phi(r)\,r^n\,, \quad r \in R_+ \;. \tag{1}$$

Note that

$$\Phi \in \Im_n \quad \Rightarrow \quad \int_0^r \Phi(\rho)\,\rho^{n-1}\,d\rho \cong \Phi(r)\,r^n\,, \quad r \in R_+ \;. \tag{2}$$

$$A \cong B \quad \Leftrightarrow \quad \exists\, c \in [1, \infty): \quad c^{-1} \le A / B \le c ,$$

For $\Phi \in \mathfrak{I}_n$, set

$$\varphi(t) = \Phi\left(t^{1/n} \right) \quad , \; t \in R_+ . \tag{3}$$

Note that

$$\Phi \in \mathfrak{I}_n \quad \Rightarrow \quad \varphi \in \mathfrak{I}_1 \quad \Rightarrow \quad \int_0^t \varphi(\tau)\, d\tau \cong \varphi(t)\, t , \quad t \in R_+ \; .$$

Introduce kernels $G: \quad R^n \to R^1$. We assume that

$$G(x) \cong \Phi(\rho), \quad \rho = |x|, \quad x \in R^n ; \quad \Phi \in \mathfrak{I}_n \; . \tag{4}$$

$$G^{*} \cong G^{**} \cong \varphi \tag{5}$$

Here f^{*} is a *decreasing rearrangement* of f

$$f^{**}(t) = t^{-1}\int_{0}^{t} f^{*}(\tau)\,d\tau, \quad t \in R_{+}. \tag{6}$$

$$H_{p}^{G}(R^{n}) = \{u = G * f : \quad f \in L_{p}(R^{n})\}. \tag{7}$$

Definition 1. *For*

$$\Phi \in \Im_{n}, \ 1 \le p < \infty, \qquad \varphi \in L_{p'}(1,\infty), \ 1/p + 1/p' = 1,$$

the potentials (7) with kernels (4) are called Riesz- type potentials.

Classical Riesz potentials: $G(x) = \rho^{\alpha - n}$, $\rho = |x|$ satisfy this definition with $\Phi(\rho) = \rho^{\alpha - n}$, $0 < \alpha < n / p$ ([1; Ch. 5]).

$$u(x) = (G * f)(x) = \int_{R^n} G(x - y) f(y) dy, \qquad (8)$$

$$\| u \|_{H_p^G} := \inf \left\{ \| f \|_{L_p} : \quad f \in L_p(R^n); \ G * f = u \right\} \qquad (9)$$

(factor- norm). $H_p^G(R^n)$ is a Banach space, and

$$H_p^G(R^n) \subset L_p(R^n) + L_\infty(R^n).$$

The problem: *to find sharp and constructive conditions of the embedding*

$$H_p^G(R^n) \subset X(R^n), \tag{10}$$

into an RIS $X(R^n)$ *and describe the optimal (the smallest) RIS* $X_0 = X_0(R^n)$ *for the embedding (10)* , i. e., such an RIS that (10) take place for $X = X_0$, and if (10) holds for some RIS X then $X_0 \subset X$ (X_0 is called rearrangement invariant hull for the space of potentials). Here RIS is a BFS with the norm monotone with respect to decreasing rearrangements. Concept covers spaces L_p, Lorentz, Marcinkiewicz and Orlicz spaces.

Luxemburg representation: for RIS $X(R^n)$ there exists unique RIS $\tilde{X}(R_+)$ such that

$$\| f \|_{X(R^n)} = \| f^* \|_{\tilde{X}(R_+)} . \qquad (11)$$

In [4, 5] the problem (10) was reduced to the description of properties of *combined Hardy operators*

$$\Re(g;t) = \int_0^\infty \varphi(\max(t;\tau))\, g(\tau)\, d\tau : \quad L_p(R_+) \to \tilde{X}(R_+); \qquad (12)$$

$$\Re(g;t) = \varphi(t) \int_0^t g(\tau)\, d\tau + \int_t^\infty \varphi(\tau) g(\tau)\, d\tau .$$

2. OPTIMAL EMBEDDINGS FOR $p = 1$.

Let $\varphi \in \Im_1$. Introduce Marcinkiewicz space $M_\varphi(R^n)$:

$$\| f \|_{M_\varphi} := \sup_{t>0} \left[f^{**}(t) \varphi(t)^{-1} \right] \cong \sup_{t>0} \left[f^{*}(t) \varphi(t)^{-1} \right]. \tag{13}$$

Theorem 1. *A). For every RIS* $X(R^n)$ *the following equivalence holds*

$$H_1^G(R^n) \subset X(R^n) \iff \varphi \in \tilde{X}(R_+). \tag{14}$$

B). The optimal RIS for embedding is $X_0(R^n) = M_\varphi(R^n)$ *with the norm (13).*

Example 1. For classical Riesz potentials we have $\varphi(t)=t^{\alpha/n-1}$ in (13) and (14). For example, if $X(R^n)=L_q(R^n)$, we have $\widetilde{X}(R_+)=L_q(R_+)$, so that condition (14) is violated for each $q\in[1,\infty]$.

Example 2. RIS are weighted Lorentz spaces $\Lambda_q(w)$, $\Gamma_q(w)$, where $w>0$ is weight, $1\le q\le\infty$,

$$\| f \|_{\Lambda_q(w)} = \| f^* w \|_{L_q(R_+)}; \qquad \| f \|_{\Gamma_q(w)} = \| f^{**} w \|_{L_q(R_+)}. \tag{15}$$

Then,

$$H_1^G(R^n) \subset \Lambda_q(w) \Leftrightarrow H_1^G(R^n) \subset \Gamma_q(w) \Leftrightarrow \| \varphi w \|_{L_q(0,T)} < \infty$$

3. OPTIMAL EMBEDDINGS FOR $1 < p < \infty$

By [3; Theorem 5], for $1 < p < \infty$, $1/p + 1/p' = 1$:

$$H_p^G(R^n) \subset L_\infty(R^n) \Leftrightarrow \varphi \in L_{p'}(R_+),$$

Let now,

$$\varphi \in \mathfrak{I}_1 \cap L_{p'}(t,\infty),\ t \in R_+;\quad \varphi \notin L_{p'}(R_+). \tag{16}$$

$$V(t) := \varphi(t)^{p'-1}\left(\int_t^\infty \varphi^{p'}\,d\tau\right)^{-1},\quad t \in R_+. \tag{17}$$

Theorem 2. *Let* $1 < p < \infty,$ *, and conditions (16) be fulfilled. Then the optimal RIS for the embedding*

$$H_p^G(R^n) \subset X(R^n)$$

has equivalent norm

$$\| f \|_{X_0} \cong \| f \|_{\Gamma_p(V)} := \left(\int_0^\infty [f^{**}(t) V(t)]^p \, dt \right)^{1/p}. \quad (18)$$

Moreover, under these assumptions $\Gamma_p(V) = \Lambda_p(V)$ *so that we obtain an equivalent norm with replacing in (18)* f^{**} *by* f^**.*

Remark 1. If φ satisfies not only (16) but also

$$\int_t^\infty \varphi^{p'} d\tau \cong \varphi(t)^{p'} t \ , \quad t \in R_+ \ ,$$

(*non-limiting case of embedding*) then, in (17)-(18)

$$V(t) \cong [\varphi(t) t]^{-1} \ , \ t \in R_+ \ .$$

Example 3. For classical Riesz potentials $\varphi(t) = t^{\alpha/n-1}$, $0 < \alpha < n/p$, and Remark 1 gives (18) with $V(t) \cong t^{-\alpha/n}$ (equivalent norm with replacing here f^{**} by f^{*}). For $\alpha \geq n/p$ space of potentials is not embedded in any RIS.

$$E_{pq} = \sup_{t \in R_+} \left[\left(\int_0^t w^q \, d\tau \right)^{1/q} \left(\int_t^\infty \varphi^{p'} \, d\tau \right)^{1/p'} \right], \quad p \le q;$$

$$E_{pq} = \left\{ \int_0^\infty [\ldots]^r \frac{\varphi(t)^{p'} \, dt}{\int_t^\infty \varphi^{p'} \, d\tau} \right\}^{1/r}, \quad p > q; \quad r = \frac{pq}{p-q} .$$

Theorem 3. *Let* $1 < p, q < \infty$, *and the conditions (16) be fulfilled. Then, the following equivalences hold:*

$$H_p^G(R^n) \subset \Gamma_q(w) \iff H_p^G(R^n) \subset \Lambda_q(w) \iff E_{pq} < \infty,$$

Remark 4. Under assumptions of Theorem 3, let at least one of the following additional conditions a) or b) be fulfilled:

a) $\int_t^\infty \varphi^{p'} d\tau \cong \varphi(t)^{p'} t \;, \quad t \in R_+ \;;$

b) $\exists\, \varepsilon > 0: \quad t^{-\varepsilon} \left(\int_0^t w^q \, d\tau \right)^{1/q}$ is increasing .

Then,

$$E_{pq} \cong \sup_{t \in R_+} \left[\left(\int_0^t w^q \, d\tau \right)^{1/q} t^{1/p'} \varphi(t) \right], \quad p \le q \,;$$

$$E_{pq} \cong \left\{ \int_0^\infty \left[\left(\int_0^t w^q \, d\tau \right)^{1/q} t^{1/p'} \varphi(t) \right]^r \frac{dt}{t} \right\}^{1/r}, \quad p > q \,.$$

Example 4. We obtain the criterion of embedding into $L_q(R^n)$, $1 \le q < \infty$, if we put here $w = 1$, so that condition b) is satisfied, and

$$H_p^G(R^n) \subset L_q(R^n) \Leftrightarrow \{ p < q \; ; \; E_{pq} < \infty \},$$

where

$$E_{pq} \cong \sup_{t>0} [\, t^{1/q+1/p'} \varphi(t) \,].$$

For the classical Riesz potentials $\varphi(t) = t^{\alpha/n-1}$, $0 < \alpha < n$, so that

$$H_p^G(R^n) \subset L_q(R^n) \Leftrightarrow \{ p < q ; \; \alpha = n(1/p - 1/q) \}.$$

REFERENCES

1. Stein E. *Singular integrals and differentiability properties of functions*. Princeton, New Jersey, 1970.
2. Bennett C., Sharpley R. // Pure and Appl. Math. 1988. V. 129.
3. Goldman M. // Dokl. Math. 2009. V. 423, N1. pp. 14-18.
4. Goldman M. // Complex Variables and Elliptic Equations. 2010. V. 55, pp. 817-832
5. Goldman M. // Proc. Steklov Inst. Math., 2010. V. 269, pp. 91-111.
6. Sawyer E. // Studia Math. 1990. V. 96. pp. 145-158.

7. Goldman M. // Proc. Steklov Inst. Mathem. 2001. V. 232. pp. 115-143.
8. Carro M., Soria J., Pick L., Stepanov V. // Math. Ineq. Appl. 4 (2001), pp. 397-428.
9. Gogatishvili A., Pick L. // Publ. Mat. 2003. V. 47. pp. 311-358.
10. Gogatishvili A., Johansson M., Okpoti C. A., and Persson L.-E. // Bull. Austral. Math. Soc., 76 (2007), pp. 69-92.
11. Goldman M. // Doklady Math. 2007. V. 414, N2. pp. 159-164.