Besov Regularity for (Nonlinear) Elliptic Boundary Value Problems

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DFG SPP 1324

AG Numerik und Optimierung

Outline

Motivation:

- Numerical treatment of elliptic boundary value problems

$$\begin{aligned} L(u) &= f \quad \text{in} \quad \mathcal{O} \subset \mathbb{R}^d \qquad \text{Lipschitz} \\ u &= 0 \quad \text{on} \quad \partial\mathcal{O} \end{aligned}$$

$$L : H_0^{m}(\mathcal{O}) \longrightarrow H^{-m}(\mathcal{O})$$

- example:

$$\begin{aligned} -\triangle u &= f \quad \text{in} \quad \mathcal{O} \subset \mathbb{R}^d \qquad \text{Lipschitz} \\ u &= 0 \quad \text{on} \quad \partial\mathcal{O} \end{aligned}$$

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Numerical scheme:

- Galerkin approach

$$\dots S_{j-1} \subset S_j \subset S_{j+1} \subset \dots,$$

$$S_j = \text{span}\{\zeta_1, \dots, \zeta_{n_j}\}, \qquad \text{dim} S_j = n_j < \infty$$

$$\langle L u_j, v \rangle = \langle f, v \rangle \quad \text{for all} \quad v \in S_j$$

$$\mathbf{A}_j \mathbf{c}_j = \mathbf{F}_j, \qquad \mathbf{A}_j = (\langle L\zeta_{l'}, \zeta_l \rangle)_{l,l'=1,\dots,n_j}$$

$$\mathbf{F}_j = (\langle f, \zeta_l \rangle)_{l=1}^{n_j}$$

$$u_j = \sum_{l=1}^{n_j} (\mathbf{c}_j)_l \zeta_l$$

finite elements, wavelets...

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Adaptive methods:

- updating strategy
  - S_j depends on u
  - self–regulating
- Question : Do we gain efficiency?
- When does adaptivity pay?

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Adaptivity

Besov Regularity for Linear Problems

Besov Regularity for Nonlinear Problems

- Fix Points for Nonlinear Operators in Quasi-Banach Spaces
- Compositions of Functions
- Besov Regularity

Wavelets:

- Multiresolution Analysis $\{V_j\}_{j\geq 0}$

$$V_0 \subset V_1 \subset V_2 \subset \dots \qquad \overline{\bigcup_{j=0}^{\infty} V_j} = L_2(\mathcal{O})$$

$$V_j = \overline{\text{span}\{\varphi_{j,k},\ k \in I_j\}}$$

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$$V_{j+1} = V_j \oplus W_{j+1} \quad V_0 = W_0 \quad L_2(\mathcal{O}) = \oplus_{j=0}^{\infty} W_j$$

$$W_j = \overline{\text{span}\{\psi_{j,k},\ k \in \mathcal{J}_j\}}$$

- in addition: norm equivalences, cancellation property, locality

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When does adaptivity pay?

- nonadaptive schemes

$$E_j(u) = \inf_{g\in V_j} \|u-g\|_{L_p(\mathcal{O})} \lesssim 2^{-\alpha j}|u|_{W^{\alpha}(L_p(\mathcal{O}))}$$

$$E_j(u) = O(n_j^{-\alpha/d}) \Longleftarrow u \in W^{\alpha}(L_p(\mathcal{O}))$$

- adaptive schemes

"ideal" algorithm: best n–term approximation

$$\mathcal{M}_n := \{g = \sum_{(j,k)\in\Lambda} c_{j,k}\psi_{j,k} \mid |\Lambda| = n\}$$

$$\sigma_n(u)_{L_p(\mathcal{O})} := \inf_{g\in\mathcal{M}_n} \|u-g\|_{L_p(\mathcal{O})} \sim \|u-g_n\|_{L_p(\mathcal{O})}$$

$$g_n = \sum_{(j,k)\in\Lambda_n} d_{j,k}\psi_{j,k}, \quad \Lambda_n \mathrel{\widehat{=}} n \text{ biggest wavelet coefficients}$$

$$\sigma_n(u)_{L_p(\mathcal{O})} = O(n^{-\alpha/d}) \Leftarrow u \in B^{\alpha}_{\tau}(L_\tau(\mathcal{O})) \quad \frac{1}{\tau} = \left(\frac{\alpha}{d} + \frac{1}{p}\right)$$

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Natural question:

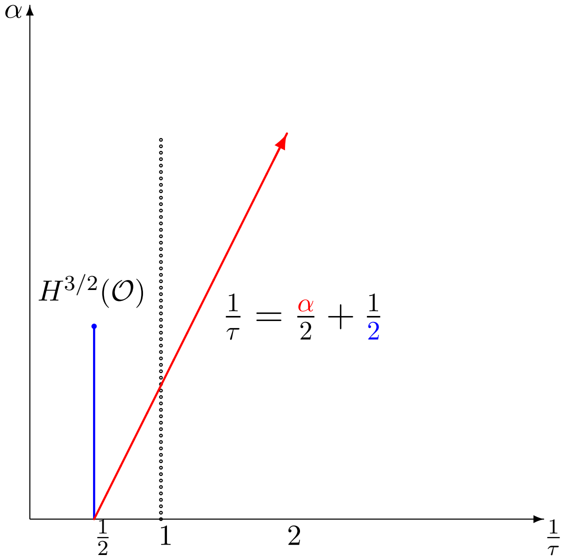
$$u \in B^{\alpha}_{\tau}(L_\tau(\mathcal{O})),\ 0 < \alpha < \alpha^* \ ?$$

Poisson equation

$$\begin{aligned} -\triangle u &= f \quad \text{in} \quad \mathcal{O} \subset \mathbb{R}^d \quad \text{Lipschitz}, \\ u &= 0 \quad \text{on} \quad \partial\mathcal{O} \end{aligned}$$

$$f \in L_2(\mathcal{O}) \Longrightarrow u \in H^{3/2}(\mathcal{O})!$$

The DeVore-Triebel diagram, $d = 2$:



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The problem:

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\end{aligned}
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Question: $f \in H_p^{t-1}(\mathcal{O})$, Bessel potential space $\Longrightarrow$

$u \in B_\tau^\alpha(L_\tau(\mathcal{O})), \quad \alpha < \alpha^*?$

The result:

Theorem (D./DeVore/Jerison/Kenig)

(i) *Let* $1 < p < p_0'$, $t \geq 1/p$, $f \in H_p^{t-1}(\mathcal{O}) \Longrightarrow u$ *belongs to all spaces* $B_\tau^{\alpha-\varepsilon}(L_\tau(\mathcal{O}))$, *where*

$$\left(\frac{1}{\tau}, \alpha\right) \in \left(\left\{\left(\frac{1}{q}, \beta\right) : \beta \leq \min(t+1, 1+\frac{1}{q}), \frac{d-1}{d+1} < q \leq p\right\} \cup \left\{\left(\frac{1}{q}, \beta\right) : \beta \leq \min\left(t+1, \frac{2d}{d-1}\right), 0 < q \leq \frac{d-1}{d+1}\right\}\right).$$

(ii) *Let* $p_0' \leq p < \infty$ *and* $t + 1 \geq 1 + \frac{1}{p_0'}$. *Then*

$$\left(\frac{1}{\tau}, \alpha\right) \in \left(\left\{\left(\frac{1}{q}, \beta\right) : \beta \leq \min(t+1, 1+\frac{1}{q}), \frac{d-1}{d+1} < q \leq p_0'\right\} \cup \left\{\left(\frac{1}{q}, \beta\right) : \beta \leq \min\left(t+1, \frac{2d}{d-1}\right), 0 < q \leq \frac{d-1}{d+1}\right\}\right).$$

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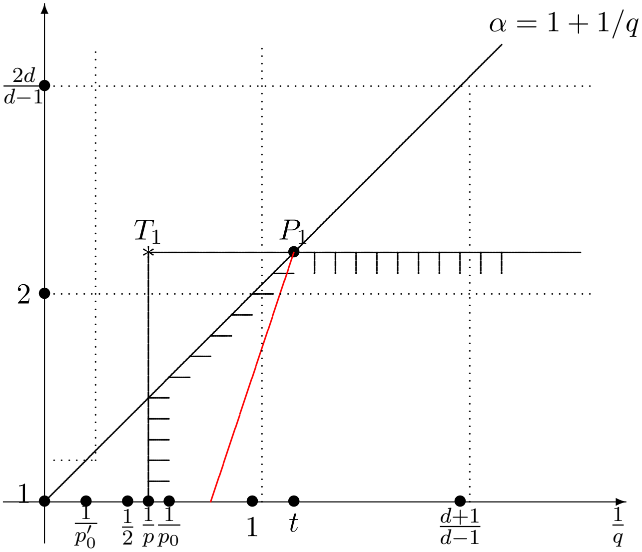
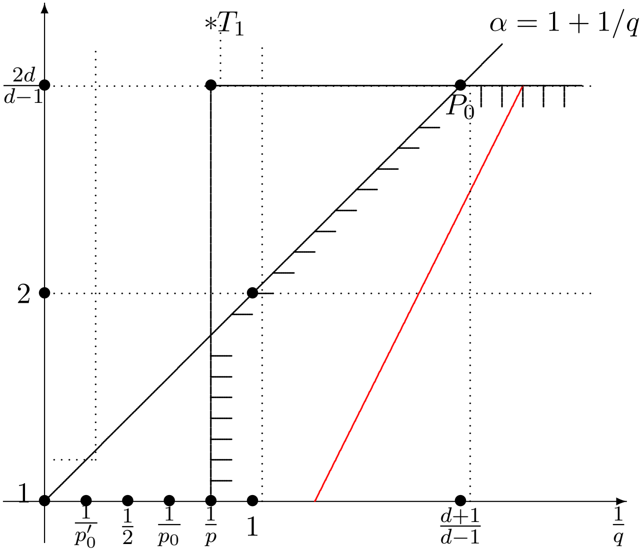


$$T_1 := (1/p, t{+}1),\ u \in B^{\alpha}_{\tau}(L_{\tau}(\mathcal{O}))\,.\ \tau := \frac{1}{t},\ \alpha < 1{+}1/\tau = 1{+}t$$



$$u \in B^{\alpha}_{\tau}(L_\tau(\mathcal{O}))\,, \quad \tau := \frac{d-1}{d+1}\,, \quad \alpha < 1 + 1/\tau = \frac{2d}{d-1}$$

Important consequences:

$$\mathcal{B}^t_p(L_p(\mathcal{O})) := \left\{ v \in B^t_p(L_p(\mathcal{O})) : \quad \mathrm{tr}\, v = 0 \right\}$$

Corollary

Let $1 < p < \infty$ *and* $\varepsilon > 0$. *Let* $L_0 = -\triangle^{-1}$ *be the solution operator of the Poisson problem, i.e.,* $L_0 f = u$.

- *Let* $t = 1$. *Then*

$$L_0 \in \mathcal{L}\Big(L_p(\mathcal{O}), \mathcal{B}^{2-\varepsilon}_1(L_1(\mathcal{O}) \Big).$$

- *Let* $1 < t < (d+1)/(d-1)$. *Then*

$$L_0 \in \mathcal{L}\Big(H^{t-1}_p(\mathcal{O}), \mathcal{B}^{1+t-\varepsilon}_{1/t}(L_{1/t}(\mathcal{O})) \Big).$$

- *Let* $t \geq (d+1)/(d-1)$. *Then*

$$L_0 \in \mathcal{L}\Big(H^{t-1}_p(\mathcal{O}), \mathcal{B}^{\frac{2d}{d-1}-\varepsilon}_\tau(L_\tau(\mathcal{O})) \Big), \qquad \tau = \frac{d-1}{d+1}.$$

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Nonlinear problems?

- $$\begin{aligned} -\triangle u(x) + g(x, u(x)) &= f(x) \quad \text{in} \quad \mathcal{O} \\ u(x) &= 0 \quad \text{on} \quad \partial\mathcal{O} \end{aligned} \qquad (1)$$

  Caratheodory condition

  - $\forall \xi \in \mathbb{R}$ $x \mapsto g(x, \xi)$ is Lebesgue measurable on $\mathcal{O}$.
  - For almost all $x \in \mathcal{O}$ $\xi \mapsto g(x, \xi)$ is continuous on $\mathbb{R}$.

- Conjecture: g "small" $\Longrightarrow$ same regularity as for the linear case

- Idea: fixed point problem

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Fixed point theorems:

Definition

A is *admissible*, if for every compact subset $K \subset A$ and for every $\varepsilon > 0$ $\exists$ continuous map $T : K \to A$ such that $T(K)$ is contained in a finite-dimensional subset of A and $x \in K$ implies $\| Tx - x | A \| \leq \varepsilon$.

Setting:

- X, Y admissible quasi-Banach spaces.
- Assume that $L : Y \to X$ is a linear and continuous operator
- $N : X \to Y$ is (in general) a nonlinear map.
- Find a fix point of the problem

$$u = (L \circ N)u. \qquad (2)$$

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Proposition (Fučík/Runst/Sickel)

Suppose, $\exists\ \eta \geq 0$, $\vartheta \geq 0$ *and* $\delta \geq 0$ *such that*

$$\| Nu|Y \| \leq \eta + \vartheta \, \| u|X \|^{\delta}.$$

Assume that $L \circ N : X \to X$ *is completely continuous* $\Longrightarrow \exists$ *solution* $u \in X$ *of* (2) *provided that*

(a) $\delta \in [0,1)$,

(b) $\delta = 1,\ \vartheta < \| L \|^{-1}$,

(c) $\delta > 1$ *and* $\eta \, \| L \| < \left[\frac{1}{\vartheta \, \| L \|} \right]^{\frac{1}{\delta - 1}} \left[\left(\frac{1}{\delta}\right)^{\frac{1}{\delta-1}} - \left(\frac{1}{\delta}\right)^{\frac{\delta}{\delta-1}} \right]$.

We are in good luck!

Proposition

Let $\mathcal{O}$ be a bounded Lipschitz domain.

- $H_p^t(\mathcal{O})$ *are admissible.*
- $B_q^t(L_p(\mathcal{O}))$ *are admissible.*
- *Let* $0 < p, q \leq \infty$ *and*

$$t > \frac{1}{p} + (d-1)\left(\frac{1}{p} - 1\right)_+.$$

Then $\mathcal{B}_q^t(L_p(\mathcal{O}))$ *are admissible.*

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Let us start...

Setting:

- $$\begin{aligned} -\triangle u(x) + g(x, u(x)) &= f(x) \quad \text{in} \quad \mathcal{O} \\ u(x) &= 0 \quad \text{on} \quad \partial\mathcal{O} \end{aligned}$$
- $g \in \mathrm{Car}(\mathcal{O} \times \mathbb{R})$,

$$|g(x,\xi)| \leq a + b\,|\xi|^{\delta},\ a, b, \geq 0,\ 0 \leq \delta \leq 1,\ x \in \mathcal{O},\ \xi \in \mathbb{R}.$$

- $$Nu(x) := f(x) - g(x, u(x)), \qquad x \in \mathcal{O}.$$
- $$X := B^{\alpha}_{\tau}(L_{\tau}(\mathcal{O})), \qquad Y := L_p(\mathcal{O}), H^t_p(\mathcal{O})$$
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Mapping properties of N, case $L_p(\mathcal{O})$:

Lemma

Let $f \in L_p(\mathcal{O})$. *If*

$$\alpha > d \max\left(0, \frac{1}{\tau} - \frac{1}{\max(1, \delta\, p)}\right),$$

then N *is continuous and bounded s.t.*

$$\| Nu \|_{L_p(\mathcal{O})} \leq \eta + \vartheta \| u \|^{\delta}_{B^{\alpha}_{\tau}(L_\tau(\mathcal{O}))}$$

where

$$\eta := \| f \|_{L_p(\mathcal{O})} + a\, |\mathcal{O}|^{1/p}, \vartheta := b \| I \,|\mathcal{L}(B^{\alpha}_{\tau}(L_\tau(\mathcal{O})), L_{\delta p}(\mathcal{O}))\|^{\delta}.$$

Mapping properties of N, case $H_p^t(\mathcal{O})$:

additional conditions:

$$
\begin{aligned}
|g(x,\xi)| &\leq a + b|\xi|^\delta \\
|g(x,\xi) - g(x,\eta)| &\leq c_1\, |\xi - \eta|^\delta \\
|g(x,\xi) - g(y,\xi)| &\leq c_2\, |x - y|^\delta
\end{aligned}
$$

Lemma

Assume

$$
d\,\max\left(0, \frac{1}{p} - \delta, \frac{1}{2} - \delta\right) < t < \delta, \;\; \alpha > \frac{t}{\delta} + d\,\max\left(0, \frac{1}{\tau} - \frac{1}{p\,\delta}\right).
$$

Let $f \in H_p^t(\mathcal{O}) \Longrightarrow N$ *is bounded and continuous with*

$$
\| Nu \|_{H_p^t(\mathcal{O})} \leq \eta^* + \vartheta^* \, \| u \|^\delta_{B^\alpha_\tau(L_\tau(\mathcal{O}))} .
$$

Harvest: Case $L_p(\mathcal{O})$:

Theorem

Let $\mathcal{O}$ be a bounded Lipschitz domain in $\mathbb{R}^d$, $0 < \delta < 1$. Then for any $f \in L_p(\mathcal{O})$ there exists at least one solution $u \in B_1^\alpha(L_1(\mathcal{O}))$, $\alpha < 2$.

Idea of Proof: We have

- admissible spaces
- bounded and continuous N
- bounded operator $L_0 = \triangle^{-1}$

Combine with fixed point theorem $\Longrightarrow$ q.e.d

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Let $\mathcal{O}$ be a bounded Lipschitz domain in $\mathbb{R}^d$, $0 < \delta < 1$. Then for any $f \in L_p(\mathcal{O})$ there exists at least one solution $u \in B_1^\alpha(L_1(\mathcal{O}))$, $\alpha < 2$.

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Harvest: The case $H^t_p(\mathcal{O})$:

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Let $\delta < 1$ *and* $1 < t < (d+1)/(d-1)$. *Further we assume*

$$1 + d \max\left(0, \frac{1}{p} - \delta, \frac{1}{2} - \delta\right) < t < \delta + 1$$

and

$$\frac{t-1}{\delta} + d \max\left(0, t - \frac{1}{p\,\delta}\right) < \alpha < t + 1 \,.$$

Then for $f \in H^{t-1}_p(\mathcal{O})$ $\exists$ *at least one solution* $u \in B^{\alpha}_{1/t}(L_{1/t}(\mathcal{O}))$.

$\delta = 1$:

Theorem

Let $\delta = 1$. *Further we assume*

$$dt - \frac{d}{p} + t - 1 < \alpha < t + 1\,.$$

If c_1, c_2 *and* b *are sufficiently small, then for any* $f \in H_p^{t-1}(\mathcal{O})$ *there exists at least one solution* $u \in B^{\alpha}_{1/t}(L_{1/t}(\mathcal{O}))$.

Some remarks:

- $\delta > 1$ possible, but more difficult (work in progress)
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Let $d = 3$ and $\delta < 1$. If $f \in L_2(\mathcal{O})$ then

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with ϵ arbitrary small.

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A typical result:

Approximation in $H^1(\mathcal{O})$:

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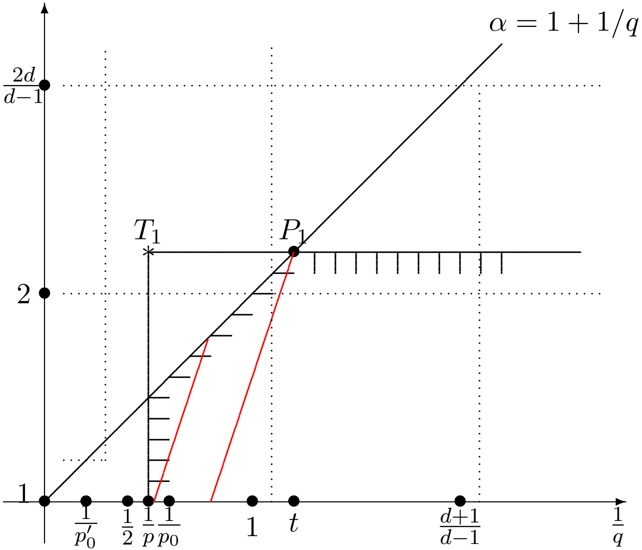
$\mathbb{R}^2 \supset \mathcal{O}$ *bounded, Lipschitz,* $f \in H_2^1(\mathcal{O})$

$$\left.\begin{array}{rl} \triangle u & = f, \\ u|_{\partial \mathcal{O}} & = 0 \end{array}\right\} \Longrightarrow v \in B_\tau^\alpha(L_\tau(\mathcal{O})), \qquad \alpha < 2$$

$1/\tau = (\alpha - 1)/2 + 1/2$

The DeVore-Triebel diagram



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- adaptive numerical schemes for (linear and nonlinear) elliptic PDE's
- theoretical analysis
- Besov regularity
  - $u \in B^s_\tau(L_\tau(\mathcal{O}))$, $1/\tau = s/d + 1/p$, $0 < s < s^*$?
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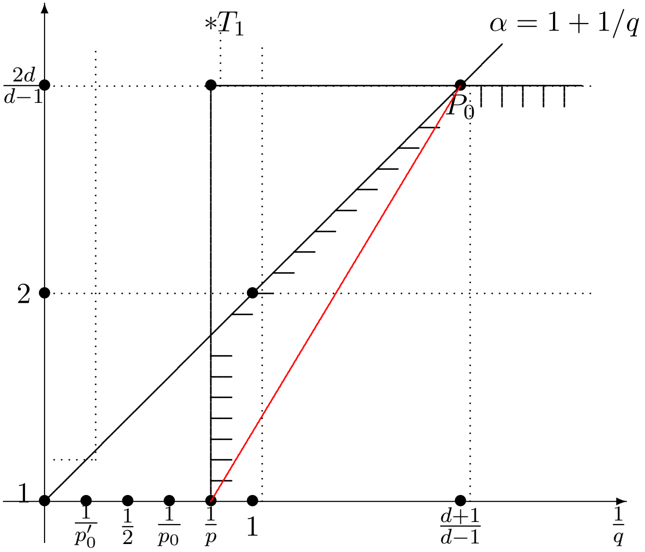

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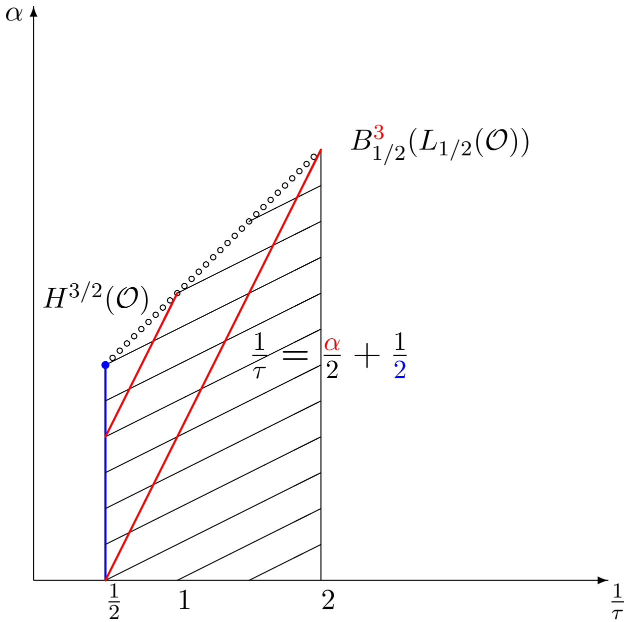

Regularity spaces, Dirichlet problem, $g \in B^1_2(L_2(\partial\mathcal{O}))$