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The approximation by Vallee Poussin sums

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I. The approximation on the classes 2π-periodic functions

Let L denote the space of integrable 2π-periodic functions $f \in L$, and let

$$S[f] = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^{\infty} A_k(f; x)$$

be the Fourier series of a function f. Further, let $\bar{\psi} = (\psi_1; \psi_2)$ be a pair of arbitrary fixed systems of numbers $\psi_1(k)$ and $\psi_2(k)$, $k = 1, 2, ...$ Consider the following series:

$$A_0 + \sum_{k=1}^{\infty}(\psi_1(k)A_k(f; x) + \psi_2(k)\tilde{A}_k(f; x)), \quad (1)$$

where A_0 is a certain number and

$$\tilde{A}_k(f; x) = a_k \sin kx - b_k \cos kx.$$

If series (1), for a given function $f(\cdot)$ and a pair $\bar{\psi}$, is the Fourier series of a certain function $F \in L$, then we say (following Prof. A. Stepanets) that F is the $\bar{\psi}$-integral of the function f $(F(\cdot) = \mathcal{I}^{\bar{\psi}}(f; \cdot))$.

We denote the set of $\bar{\psi}$-integrals of all functions $f \in L$ by $L^{\bar{\psi}}$. If $\mathfrak{N}$ is a subset of L, then $L^{\bar{\psi}}\mathfrak{N}$ denotes the set of $\bar{\psi}$-integrals of functions $f \in \mathfrak{N}$.

Then, if C is the space of continuous 2π-periodic functions, we denote

$$C^{\bar{\psi}} = L^{\bar{\psi}} \cap C, \quad C^{\bar{\psi}}\mathfrak{N} = L^{\bar{\psi}}\mathfrak{N} \cap C.$$

Let M denote a subset of functions $f \in L$ with the finite norm $\|\cdot\|_M$, where $\|\varphi\|_M = \operatorname{ess\,sup}|\varphi|$. Then S_M is the unit ball in the space M:

$$S_M = \{\varphi : \quad \|\varphi\| \leq 1\}.$$

If $\mathfrak{N} = S_M$, then we denote

$$C^{\bar{\psi}} S_M \stackrel{\text{df}}{=} C^{\bar{\psi}}_{\infty}.$$

Further, let H_ω is the class of functions $f \in C$ satisfying the condition

$$|f(x) - f(x')| \leq \omega(|x - x'|),$$

where $\omega(t)$ is a certain modulus of continuity. Then if $\mathfrak{N} = H_\omega$, we will write $f \in C^{\bar{\psi}} H_\omega$.

If $\psi_1(n) = \psi(n)\sin\frac{\beta\pi}{2}$, $\psi_2(n) = \psi(n)\cos\frac{\beta\pi}{2}$, then

$$C^{\bar\psi} = C^{\psi}_{\beta} \qquad (A.\ Stepanets, 1983)$$

If $\psi_1(n) = n^{-r}\sin\frac{\beta\pi}{2}$, $\psi_2(n) = n^{-r}\cos\frac{\beta\pi}{2}$, $r > 0$ then

$$C^{\bar\psi}_{\infty} = W^r_{\beta}, \quad C^{\bar\psi}H_{\omega} = W^r_{\beta}H_{\omega} \quad (Weyl - Nagy)$$

If $\psi_1(n) = n^{-r}\sin\frac{r\pi}{2}$, $\psi_2(n) = n^{-r}\cos\frac{r\pi}{2}$, $r \in \mathbb{N}$ then

$$C^{\bar\psi}_{\infty} = W^r.$$

Without loss of generality, we assume that the sequences $\psi(k)$ from the set $\mathfrak{M}$ are restrictions of certain positive continuous convex downwards functions $\psi(t)$ of continuous argument $t \geq 1$ that vanish at infinity to the set of natural numbers:

$$\mathfrak{M} = \{\psi(t) : \quad \psi(t) > 0,$$
$$\psi(t_1) - 2\psi((t_1 + t_2)/2) + \psi(t_2) \geq 0$$
$$\forall t_1, t_2 \in [1; \infty), \quad \lim_{t \to \infty} \psi(t) = 0\}.$$

Let $\psi \in \mathfrak{M}$. Then we denote by $\eta(t) = \eta(\psi; t)$ the function connected with ψ by the equality

$$\psi(\eta(t)) = \frac{1}{2}\psi(t), \quad t \geq 1. \qquad (2)$$

The function $\mu(t)$ is defined by the equality

$$\mu(t) = \mu(\psi; t) = \frac{t}{\eta(t) - t}.$$

As follows from (2), the quantity $\eta(t) - t$ is the length of the segment $[t; \eta(t)]$ on which the value of the function ψ reduces exactly to half. In this connection, the function $\mu(\psi; t)$ was called the modulus of half-decay of the function ψ.

The quantity $\mu(\psi; t)$ can be bounded from above and below by certain positive numbers, can tend to zero as $t \to \infty$, and can be unbounded above. On the basis of these features, we (following Prof. A. Stepanets) select the following subsets of the set $\mathfrak{M}$:

$$\mathfrak{M}_0 = \{\psi \in \mathfrak{M} : \quad 0 < \mu(\psi; t) \leq K \quad \forall t \geq 1\},$$

$$\mathfrak{M}_\infty = \{\psi \in \mathfrak{M} : \; 0 < K \leq \mu(\psi; t) < \infty \; \forall t \geq 1\},$$

$$\mathfrak{M}_C = \{\psi \in \mathfrak{M} : \; 0 < K_1 \leq \mu(\psi; t) \leq K_2 \; \forall t \geq 1\}.$$

$$F = \{\psi \in \mathfrak{M} : \quad \eta'(\psi; t) \leq K\}.$$

Note that the functions t^{-r}, $r > 0$, $t^{-r} \ln^{\varepsilon}(t+e)$ for $\varepsilon \in \mathbb{R}$ etc. can be regarded as natural examples of functions from the set $\mathfrak{M}_C$. The functions $\ln^{-\alpha}(t+e)$, $\alpha > 0$ are typical representatives the set $\mathfrak{M}_0 \setminus \mathfrak{M}_C$. The functions $e^{-\alpha t^r}$, $\alpha > 0$, $r > 0$ are examples of functions from the set $\mathfrak{M}_\infty$.

Further, we note results related to obtaining asymptotic equalities for the quantities

$$\mathcal{E}(C^{\bar{\psi}}\mathfrak{N}; V_{n,p}) = \sup_{f \in C^{\bar{\psi}}\mathfrak{N}} \|f(\cdot) - V_{n,p}(f;\cdot)\|_C,$$

where

$$V_{n,p}(f;x) = \frac{1}{p} \sum_{i=n-p}^{n-1} \sum_{k=0}^{i} A_k(f;x)$$

— Vallee Poussin sums of function f.

• *The rate of convergence of Vallee Poussin sums on the classes $C^{\bar\psi}$ of functions of small smoothness (V. Rukasov, S. Chaichenko)*

Suppose that $\psi_1 \in \mathfrak{M}_0'$, $\psi_2 \in \mathfrak{M}_0$. Let natural number p depends on n that the limit $\lim\limits_{n\to\infty}(p/n)$ exists and equals Θ, $0 \le \Theta < 1$.

$$\mathcal{E}(C^{\bar\psi}_\infty; V_{n,p}) = \frac{4\bar\psi(n)}{\pi^2}\ln\frac{n}{p} + \frac{2}{\pi}\int\limits_n^\infty \frac{\psi_2 t)}{t}\,dt + O(1)\bar\psi(n),$$

$$\mathcal{E}(C^{\bar\psi}H_\omega; V_{n,p}) = \theta_\omega[\frac{2\bar\psi(n)}{\pi^2}\ln\frac{n}{p}\int\limits_0^{\frac{\pi}{2}} \omega(\frac{2t}{n})\sin t\,dt +$$

$$+\frac{1}{\pi}|\int\limits_0^1 \omega(\frac{2t}{n})\int\limits_1^\infty \psi_2(nv)\sin vt\,dv\,dt|] + O(1)\bar\psi(n)\omega(1/n),$$

where $\bar\psi(n) = \sqrt{\psi_1^2(n) + \psi_2^2(n)}$, $\theta_\omega \in [\frac{2}{3}; 1]$, $\theta_\omega = 1$ if $\omega(t)$ is a convex modulus of continuity, and $O(1)$ are quantities uniformly bounded in n.

• *The rate of convergence of Vallee Poussin sums on the classes $C^{\bar\psi}$ of functions of high smoothness (V. Rukasov, S. Chaichenko)*

Suppose that $\psi_i \in F$, numbers $p = p(n)$ are chosen so that $n - p \in [\eta^{-1}(\psi_i; n); n]$, $i = 1, 2,$ and there exist constants K_1 and K_2 such that

$$0 < K_1 \le \frac{\eta(\psi_1; n) - n}{\eta(\psi_2; n) - n} \le K_2 < \infty, \quad n = 1, 2, \dots .$$

$$\mathcal{E}(C^{\bar\psi}_\infty; V_{n,p}) = \frac{4}{\pi^2}\overline{\psi}(n) \ln^+ \frac{\eta(n) - n}{p} + O(1)\overline{\psi}(n)$$

$$\mathcal{E}(C^{\bar\psi} H_\omega; V_{n,p}) = \frac{2\theta_\omega}{\pi^2}\overline{\psi}(n) \ln^+ \frac{\eta(n) - n}{p} \times$$

$$\times \int_0^{\frac{\pi}{2}} \omega(\frac{2t}{n}) \sin t \, dt + O(1)\bar\psi(n)\omega(1/n),$$

where $\overline{\psi}(n) = \sqrt{\psi_1^2(n) + \psi_2^2(n)}$, $\eta(n)$ is either $\eta(\psi_1; n)$ or $\eta(\psi_2; n)$, $\theta_\omega \in [\frac{2}{3}; 1]$, $\theta_\omega = 1$ if $\omega(t)$ is a convex modulus of continuity, and $O(1)$ are quantities uniformly bounded in n.

• *Approximation of Poisson integrals by Vallee Poussin sums (V. Rukasov, O. Novikov, S. Chaichenko)*

Let be $\psi_1(n) = q^n \sin\frac{\beta\pi}{2}$, $\psi_2(n) = q^n \cos\frac{\beta\pi}{2}$, $q \in (0;1)$, $\beta \in \mathbb{R}$.

$$\mathcal{E}(C^{\bar{\psi}}_{\infty}; V_{n,p}) = \frac{4}{\pi(1-q^2)}\frac{q^{n-p+1}}{p} +$$

$$+O(1)\left[\frac{q^n}{(1-q^2)p} + \frac{q^{n-p+1}}{(1-q)^3(n-p)p}\right],$$

$$\mathcal{E}(C^{\bar{\psi}} H_{\omega}; V_{n,p}) =$$

$$= \frac{2\theta_{\omega}}{\pi(1-q^2)}\frac{q^{n-p+1}}{p}\int\limits_0^{\pi/2}\omega(\frac{2t}{n-p})\sin t\;dt +$$

$$+O(1)\omega(\frac{1}{n-p})\left[\frac{q^n}{(1-q^2)p} + \frac{q^{n-p+1}}{(1-q)^3(n-p)p}\right],$$

where $\theta_{\omega} \in [1/2;1]$, $\theta_{\omega} = 1$ if $\omega(t)$ is a convex modulus of continuity, and $O(1)$ are quantities uniformly bounded in n.

• *Approximation by Vallee Poussin sums on classes of analytic functions (V. Rukasov, S. Chaichenko)*

Let $\psi(k) \in D_q := \{\psi(k) : \lim \frac{\psi(k+1)}{\psi(k)} = q\}$, $q \in (0;1)$, $\beta \in \mathbb{R}$, $\varepsilon_{n-p} = \sup\limits_{k \geq n-p} |\frac{\psi(k+1)}{\psi(k)} - q|$. Then for $n \to \infty$

$$\mathcal{E}(C^{\psi}_{\beta,\infty}; V_{n,p}) = \psi(n-p+1) \times$$

$$\times(\frac{4}{\pi^2}\int\limits_0^{\pi} \frac{\sqrt{1-2q^p\cos pt + q^2}}{1-2q^2\cos t + q^2}\, dt +$$

$$+O(1)\frac{q}{(1-q)^{\xi(p)}(n-p+1)} + \frac{p\varepsilon_{n-p}}{(1-q)^4}),$$

where

$$\xi(p) = \begin{cases} 1, & p = 1, \\ 3, & p = 2, 3, \ldots . \end{cases}$$

• *Approximation by Vallee Poussin sums on classes of entire functions (A. Serdyuk, E. Ovsii)*

Let $\psi(k) \in D_0$, $\beta \in \mathbb{R}$, $k, n, p \in \mathbb{N}$, $p \le n$.

$$\mathcal{E}(C^{\psi}_{\beta,\infty}; V_{n,p}) = \frac{1}{p}(\frac{4}{\pi}\psi(n-p+1)+$$
$$+O(1)(\frac{\psi^2(n-p+1)}{\psi(n-p+1)} + \gamma_3(\psi; n; p))),$$

$$\mathcal{E}(C^{\psi}_{\beta} H_{\omega}; V_{n,p}) =$$
$$= \frac{1}{p}(\frac{2\theta_\omega}{\pi}\psi(n-p+1)\int\limits_0^{\pi/2}\omega(\frac{2t}{n-p+1})\sin t\, dt+$$
$$+O(1)\omega(\frac{1}{n-p+1})\gamma_3(\psi; n; p)),$$

where

$$\gamma_m = \min\{p\sum_{k=m}^{\infty}\psi(n-p+k); \sum_{k=m}^{\infty}k\psi(n-p+k)\},$$

$\theta_\omega \in [2/3; 1]$, $\theta_\omega = 1$ if $\omega(t)$ is a convex modulus of continuity, and $O(1)$ are quantities uniformly bounded in n, p.

II. The approximation on the classes of locally integrable functions

Let $\widehat{L}_p$ be the space of functions given on the real line $\mathbb{R}$ (and not necessarily periodic), for which the norm

$$\|\varphi\|_{\widehat{p}} = \begin{cases} \sup\limits_{a\in\mathbb{R}} \Big(\int\limits_a^{a+2\pi} |\varphi(t)|^p \, dt \Big)^{1/p}, & p \in [1;\infty), \\ \operatorname*{ess\,sup}\limits_{t\in\mathbb{R}} |\varphi(t)|, & p = \infty \end{cases}$$

is finite.

Prof. A. Stepanets introduced the classes $\widehat{L}^{\psi}_{\beta}$ and L^{ψ}_{β} in the following way.

Let $\psi(t)$ be a continuous on $\{t \geq 0\}$ function and let the transform $\widehat{\psi}(t;\beta)$

$$\widehat{\psi}(t) = \widehat{\psi}(t;\beta) = \frac{1}{\pi}\int\limits_0^{\infty} \psi(v)\cos(vt + \frac{\beta\pi}{2})\,dv, \quad (3)$$

where β is a fixed real number, exists for all $t > 0$.

Let us denote by $\widehat{L}^{\psi}_{\beta}$ the set of functions $f \in \widehat{L}_1$, such that

$$f(x) = A_0 + \int\limits_{-\infty}^{\infty} \varphi(x+t)\widehat{\psi}(t)\,dt, \quad (4)$$

where A_0 is a certain constant, the integral is understood in the sense of the limit over extending symmetrical intervals (principal value), $\varphi \in \widehat{L}_1$.

If $f \in \widehat{L}^{\psi}_{\beta}$ and, furthermore, $\varphi \in \mathfrak{N}$, where $\mathfrak{N} \subset \widehat{L}_1$, then we set $f \in \widehat{L}^{\psi}_{\beta}\mathfrak{N}$. Following Prof. A. Stepanets, the function $\varphi(\cdot)$ in (4) is called the $(\psi;\beta)$-derivative of the function $f(\cdot)$ and it is denoted by $f^{\psi}_{\beta}(\cdot)$.

$$\widehat{C}^{\psi}_{\beta} := \widehat{L}^{\psi}_{\beta} \cap C.$$

We approximate functions $f \in \widehat{L}^{\psi}_{\beta}\mathfrak{N}$ by operators of the following type

$$V_{\sigma,c}(f;x) = A_0 + \int\limits_{-\infty}^{\infty} \widehat{f}^{\psi}_{\beta}(x+t)\left(\widehat{\psi\lambda}_{\sigma,c}\right)(t;\beta)\,dt,$$

where $\left(\widehat{\psi\lambda}_{\sigma,c}\right)(t;\beta)$ is the transformation of type (3) of $\psi(v)\lambda_{\sigma,c}(v)$

$$\lambda_{\sigma,c}(v) = \begin{cases} 1, & 0 \le v \le c, \\ \frac{\sigma-v}{\sigma-c}, & c \le v \le \sigma, \\ 0, & v \ge \sigma. \end{cases}$$

If $f \in C^{\psi}_{\beta}\mathfrak{N}$, where $C^{\psi}_{\beta}\mathfrak{N}$ is a subset of periodic functions of $\widehat{C}^{\psi}_{\beta}\mathfrak{N}$, $\sigma = n \in N$ and $c = n-p$, $p \in N$, $p < n$, operators $V_{\sigma;c}(f;x)$ coincide with the classical Vallee Poussin sums

$$V_{n,p}(f;x) = \frac{1}{p}\sum_{i=n-p}^{n-1}\sum_{k=0}^{i} A_k(f;x).$$

By this reason the operators $V_{\sigma,c}(f;x)$ are called Vallee Poussin operators.

We'd like to give the results of asymptotic behavior research when $\sigma \to \infty$ of the value

$$\mathcal{E}(\widehat{C}^{\psi}_{\beta}\mathfrak{N}; V_{\sigma,c}) = \sup_{f \in \widehat{C}^{\psi}_{\beta}\mathfrak{N}} \|f(x) - V_{\sigma,c}(f;x)\|_{C}.$$

Each function from $\mathfrak{M}$ is extended to the interval $[0,1)$ so, that the obtained function (which is also denoted by $\psi(\cdot)$) turns out to be continuous for all $v \geq 0$, $\psi(0) = 0$ and its derivative $\psi'(v) = \psi'(v+0)$ has bounded variation on the interval $[0, \infty)$. The set of such functions is denoted by $\mathfrak{A}$.

Then all the functions $\psi \in \mathfrak{A}$, for which with $t \geq 1$ positive numbers K_1 and K_2 are found, such as

$$K_1 \leq \frac{t}{\eta(t) - t} \leq K_2,$$

are referred to set $\mathfrak{A}_C$ and all the functions $\psi \in \mathfrak{A}$, for which

$$\eta'(t) \leq K,\ t \geq 1,\ \eta'(t) \stackrel{\mathrm{df}}{=} \eta'(t+0), K \equiv \mathrm{const} > 0,$$

are refused to set $\overline{F}$.

Theorem 1. (V. Rukasov) *Let* $\psi \in \overline{F}$. *Then there are equalities for any* $\sigma > h \geq 1$

$$\mathcal{E}(\widehat{C}^{\psi}_{\beta,\infty}; V_{\sigma,\sigma-h}) = \frac{4}{\pi^2}\psi(\sigma)\left|\ln\frac{\eta(\sigma)-\sigma}{h}\right| +$$

$$+O(1)\psi(\sigma-h),$$

$$\mathcal{E}(\widehat{C}^{\psi}_{\beta}H_{\omega}; V_{\sigma,\sigma-h}) =$$

$$= \frac{2\theta_\omega}{\pi^2}\psi(\sigma)\left|\ln\frac{\eta(\sigma)-\sigma}{h}\right| \int\limits_0^{\frac{\pi}{2}} \omega(\frac{2t}{\sigma})\sin t\; dt +$$

$$+O(1)\psi(\sigma-h)\omega(\frac{1}{\sigma-h}),$$

where O(1) is a value uniformly limited according to σ *and* β, $\theta_\omega \in [2/3; 1]$, *when* $\theta_\omega = 1$, *if* $\omega(t)$ *is a convex continuity modulus,* $h = \sigma - c$.

Theorem 2. (V. Rukasov) *Let* $\psi \in \mathfrak{A}_C$ *and* $0 < \lim\limits_{\sigma\to\infty} \frac{c}{\sigma} = \Theta < 1$. *Then with* $\sigma \to \infty$ *asymptotic equality is provided*

$$\mathcal{E}(\widehat{C}^{\psi}_{\beta,\infty}; V_{\sigma,c}) = A(\tau^{(\Theta)}_{\sigma}) + O(1)(\psi(\sigma)\varepsilon_{\sigma}), \quad (5)$$

where

$$\tau^{(\Theta)}_{\sigma}(v) = \begin{cases} 0, & 0 \le v \le \Theta, \\ \frac{v-\Theta}{1-\Theta}\psi(\sigma v), & \Theta \le v \le 1, \\ \psi(\sigma v), & v \ge 1, \end{cases}$$

$$\varepsilon_{\sigma} = \begin{cases} |\Theta - c/\sigma| \ln \frac{1}{|\Theta - c/\sigma|}, & \Theta \ne \frac{c}{\sigma}, \\ 0, & \Theta = \frac{c}{\sigma}, \end{cases} \quad (6)$$

in the case

$$A(\tau^{(\Theta)}_{\sigma}) = O(1)\psi(\sigma), \qquad \sigma \to \infty. \quad (7)$$

It follows from rations (6) – (7) that equality (5) provides the solution of Kolmogorov - Nikol'skii's problem for the operators $V_{\sigma,c}(f;x)$ on the class $\widehat{C}^{\psi}_{\beta,\infty}$. We should note that in this case the operators $V_{\sigma,c}(f;x)$ approximate functions of the classes $\widehat{C}^{\psi}_{\beta,\infty}$ in the best way possible.

Assume

$$\rho_{\sigma,c}(f;x) = f(x) - V_{\sigma,c}(f;x).$$

Let $W_\sigma^2 = \{u \in \mathcal{E}_\sigma : \int\limits_{-\infty}^{\infty} \frac{u^2(t)}{1+t^2}\,dt < \infty\}$ where $\mathcal{E}_\sigma$ — set of band-limited functions of the order at most σ and

$$E_\sigma(\varphi)_{\widehat{p}} = \inf_{u \in W_\sigma^2} \|\varphi(\cdot) - u(\cdot)\|_{\widehat{p}}$$

be the best approximation of the function φ by band-limited functions of the order at most σ.

Let

$$\psi_\alpha(v) = \begin{cases} \psi_1(v), & v \in [0;1], \\ e^{-\alpha v}, & v > 1, \end{cases}$$

where $\alpha > 0$ is an arbitrary real number, $\psi_1(v)$ is an absolutely continuous function, which has the derivative $\psi_1'(v)$ with bounded variation on the segment $[0;1]$, and such as $\psi_1(0)\sin\frac{\beta\pi}{2} = 0$ and $\psi_1(1) = e^{-\alpha}$.

$$\widehat{L}_\beta^{\psi_\alpha} = \widehat{L}_\beta^\alpha$$

• *Approximation by Vallee Poussin operators on classes $\widehat{L}^{\alpha}_{\beta}$ (V. Rukasov, S. Chaichenko)*

Let $f \in \widehat{L}^{\alpha}_{\beta}\widehat{L}_p$, $\alpha > 0$, $\beta \in \mathbb{R}$, $p \in [1;\infty]$. Then, with $\sigma \to \infty$

$$\|\rho_{\sigma,c}(f;x)\|_{\widehat{p}} \leq$$

$$\leq \frac{e^{-\alpha c}}{h}[\frac{4}{\pi^2}\int\limits_0^{\infty} \frac{\sqrt{1-2e^{-\alpha h}\cos ht + e^{-2\alpha h}}}{\alpha^2+t^2}\, dt +$$

$$+O(1)(\frac{1+\alpha}{\alpha^2\ c})]E_c(f^{\alpha}_{\beta})_{\widehat{p}}, \quad h=\sigma-c.$$

In the case $0 < K_1 \leq \sigma - c \leq K_2 < \infty$, where K_1 and K_2 are some absolute constants, the Vallee Poussin operators on classes $\widehat{L}^{\alpha}_{\beta}\widehat{L}_p$ give the approximation of the order $O(e^{-\alpha\sigma})$, $\sigma \to \infty$.

It should be noticed that last inequality is sharp on some subsets of functions. Indeed, let

$$\widehat{S}_p = \left\{ f \in \widehat{L}_p : \quad \|f\|_{\widehat{p}} \le 1 \right\}, \qquad \widehat{L}^{\alpha}_{\beta}\widehat{S}_p = \widehat{L}^{\alpha}_{\beta,p}.$$

Let $p \in [1;\infty]$, $\alpha > 0$, $\beta \in \mathbb{R}$. If $\sigma \to \infty$, the relation

$$\mathcal{E}(\widehat{L}^{\alpha}_{\beta,p}; V_{\sigma,c}) \le$$

$$\le \frac{e^{-\alpha c}}{h}[\frac{4}{\pi^2}\int\limits_0^{\infty} \frac{\sqrt{1-2e^{-\alpha h}\cos ht + e^{-2\alpha h}}}{\alpha^2+t^2}\, dt +$$

$$+ O(1)(\frac{1+\alpha}{\alpha^2\, c})], \quad h = \sigma - c,$$

is fulfilled. If $p = \infty$, it becomes an equation

$$\mathcal{E}(\widehat{L}^{\alpha}_{\beta,p}; V_{\sigma,c}) =$$

$$= \frac{e^{-\alpha c}}{h}[\frac{4}{\pi^2}\int\limits_0^{\infty} \frac{\sqrt{1-2e^{-\alpha h}\cos ht + e^{-2\alpha h}}}{\alpha^2+t^2}\, dt +$$

$$+ O(1)(\frac{1+\alpha}{\alpha^2\, c})].$$

Let us denote by $\mathfrak{A}^*$ the subset of functions of the set $\mathfrak{A}$, which for all $v \geq v_0$ have finite derivative of the second order. Define

$$\mathcal{D}_\alpha := \{\psi \in \mathfrak{A}^* : \quad \lim_{v\to\infty} \frac{\psi''(v)}{\psi'(v)} = -\alpha, \ \alpha > 0\}.$$

If $\psi \in \mathcal{D}_\alpha$, then the sequence $\psi_k = \psi(k), \ k = 1, 2, \ldots,$ is the sequence of d'Alembert

$$D_q := \{\psi_k : \lim_{k\to\infty} \frac{\psi(k+1)}{\psi(k)} = q, \ q = e^{-\alpha}, \ \alpha > 0\}.$$

Theorem 1 (V. Rukasov, S. Chaichenko, 2010). *Let* $1 \leq p \leq \infty$, $\psi \in \mathcal{D}_\alpha$. *Then for all functions* $f \in \widehat{L}^\psi_\beta \widehat{L}_p$ *and for* $c \to \infty$

$$\|\rho_{\sigma,c}(f;\cdot)\|_{\hat{p}} = \psi(c)[e^{\alpha c}\|\rho_{\sigma,c}(\mathcal{J}^\alpha_\beta(f^\psi_\beta);\cdot)\|_{\hat{p}} +$$

$$+ O(1)\frac{(\alpha^2+1)\varepsilon_c}{\alpha^3(\sigma - c)} E_c(f^\psi_\beta)_{\hat{p}}],$$

where $O(1)$ *is uniformly bounded with respect to the parameters* $\sigma, c, p, \alpha, \psi$ *i* β.

Theorem 2 (V. Rukasov, S. Chaichenko, 2010). *Let* $\psi \in \mathcal{D}_\alpha$, $\beta \in \mathbb{R}$, $1 \le p \le \infty$. *Then for* $c \to \infty$

$$\mathcal{E}(\widehat{L}^{\psi}_{\beta,p}; V_{\sigma,c})_{\hat{p}} = \psi(c)(e^{\alpha c}\mathcal{E}(\widehat{L}^{\alpha}_{\beta,p}; V_{\sigma,c})_{\hat{p}}+$$

$$+O(1)\frac{(2\alpha^2+1)}{\alpha^3(\sigma-c)}\ \varepsilon_c),$$

where $O(1)$ *is uniformly bounded with respect to the parameters* $\sigma, c, p, \alpha, \psi$ *i* β.

Corollary 1. *Under the conditions of Theorem 2 one has for an arbitrary class* $\widehat{L}^{\psi}_{\beta,p}$ *for* $c \to \infty$

$$\mathcal{E}(\widehat{L}^{\psi}_{\beta,p}; V_{\sigma,c})_{\hat{p}} \le \frac{\psi(c)}{\sigma - c}\times$$

$$\times[\frac{4}{\pi^2}\int\limits_0^\infty \frac{\sqrt{1-2e^{-\alpha(\sigma-c)}\cos(\sigma-c)t+e^{-2\alpha(\sigma-c)}}}{\alpha^2+t^2}\,dt+$$

$$+O(1)(\frac{1+\alpha}{\alpha^2\ c}+\frac{(\alpha^2+1)}{\alpha^3}\ \varepsilon_c)].$$

If $p=\infty$, *this relation becomes an equation.*

III. Approximation of holomorphic functions

Let functions f is holomorphic in a circle

$$\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$$

and

$$f(z) = \sum_{k=0}^{\infty} \widehat{f}_k z^k, \quad \widehat{f}_k := f^{(k)}(0)/k!,$$

— his division in series of Taylor.

Let further $\psi = \{\psi_k\}_{k=0}^{\infty}$, $(\psi_0 = 1)$ is a sequence of the complex number. We will define H_{∞}^{ψ} class of function f, which is holomorphic in $\mathbb{D}$ and represented in form

$$f(z) = \sum_{k=0}^{\infty} \psi_k \widehat{g}_k z^k, \quad z \in \mathbb{D},$$

where $g(z) = \sum\limits_{k=0}^{\infty} \widehat{g}_k z^k$ — holomorphic function in $\mathbb{D}$ for which $\|g\|_{\infty} := \sup\limits_{z \in \mathbb{D}} |g(z)| \leq 1$.

We have an aim to research asymptotic equation for quantity

$$R_{n,p}(H^{\psi}_{\infty}) := \sup_{f\in H^{\psi}_{\infty}} \|f - V_{n,p}(f)\|_{\infty},$$

in which

$$V_{n,p}(f)(z) := \frac{1}{p}\sum_{k=n-p}^{n-1}\left(\sum_{l=0}^{k}\widehat{f}_l z^l\right),$$

— the sums of Valle Poussin of function f.

We consider natural number p depends on n that the limit $\lim\limits_{n\to\infty}(p/n)$ exists and equals Θ.

Let think real and imaginary parts of the equation ψ are positive convex and satisfy conditions

$$\frac{\operatorname{Re}\psi_k}{\operatorname{Re}\psi_{2k}} \le K_1 < \infty, \quad \frac{\operatorname{Im}\psi_k}{\operatorname{Im}\psi_{2k}} \le K_2 < \infty\,\forall\, k\in\mathbb{N}.$$

Theorem. (V. Savchuk, S. Chaichenko, 2010) *Let* $n, p = p(n) \in \mathbb{N},\ p < n$ *and* $0 \le \Theta < 1$. *Then if* $n \to \infty$

$$R_{n,p}(H_\infty^\psi) = \frac{1}{\pi}|\psi_n| \ln \frac{n}{p} + O(1)|\psi_n|, \qquad (1)$$

where $O(1)$ *is quantity uniformly limited in respect* p *and* n.

Relation (1) is a asymptotic quantity in the cases when $\Theta = 0$.

The quantity (1) contains some well-known results.

$p = 1$ — V.V. Savchuk, 1998

$$R_{n,p}(H_\infty^\psi) = \frac{1}{\pi}|\psi_n| \ln n + O(1)|\psi_n|, \qquad (1')$$

$|\psi_n| = n^{-r},\ r > 0$ — L.V. Tajkov, 1961

$$R_{n,p}(H_\infty^\psi) = \frac{1}{\pi}\frac{1}{n^r} \ln \frac{n}{p} + O(1)\frac{1}{n^r}, \qquad (1^*)$$

$|\psi_n| = n^{-r},\ r = 1, 2, \ldots,\ p = 1$ — S.B. Stechkin, 1953

$$R_{n,p}(H_\infty^\psi) = \frac{1}{\pi}\frac{1}{n^r} \ln n + O(1)\frac{1}{n^r}, \qquad (1^\triangle)$$