V. I. Burenkov

RECENT PROGRESS IN THE PROBLEM OF SPECTRAL STABILITY ESTIMATES FOR ELLIPTIC OPERATORS

(based on joint work with P.D. Lamberti)

International Workshop on
Smoothness, Approximation, and Function Spaces
Oppurg, Germany
October 15, 2010

Plan

1. **Jena (5 min)**

2. **Astana, ENU (5 min)**

3. **EMJ (5 min)**

4. **ISAAC (5 min)**

5. **Talk (25 min)**

1. Jena

1.1. World recognised centre on Real Analysis headed by Prof. Hans Triebel.

1.2. Perfect conference organiser (Jena, Friedrichroda, Zygmundsburg, Freiburg, Oppurg etc).

1.3. Numerous joint projects (with Russia, Ukraine, Belarus, Czech Republic,...)

1.4. Nice people. Friendly atmosphere.

1.5. Sport activities, excursions etc.

1.6. MANY THANKS TO JENA.

2. Astana, Eurasian National University

2.1. New capital of Kazakhstan, modern quickly developing city.

2.2. New L.N. Gumilyov Eurasian National University (ENU) – one of the top 500 universities in the world (according to Times classification).

2.3. One of the strategies - strong encouragement and support for international cooperation.

2.4. PhD programme. Foreign consultants (M.L. Goldman, H.-J. Schmeisser, T.V. Tararykova, V.I. Burenkov and many others).

Joint supervision of PhD students. Active development of international cooperation.

In the framework of international cooperation T.V. Tararykova, V.I. Burenkov represent ENU at this Workshop.

3. Eurasian Mathematical Journal

3.1. Started in 2010, 4 issues per year.

3.2. Editors-in-chief: V.I. Burenkov, M. Otelbaev, V.A. Sadovnichy. Editorial board 50 distiguished mathematicians from many countries of Europe and Asia, including O.V. Besov and D. Haroske.

3.3. All papers in English.

3.4. International journal. More than 2/3 of papers by mathematicians from outside of Kazakhstan.

3.5. No limit on the size of a paper. Long papers are welcome. Survey papers are published from time to time.

3.6. Quick publication.

3.7. Currently honorarium is paid to the authors (50$) per page and to the reviewers (50-100 $) per review.

3.8. A paper by K. Runovsky and H.J. Schmeisser will be published in number 3 of the EMJ (2010).

4. International Society for Analysis,its Applications, and Computation (ISAAC)

4.1. Exists since 1997.

4.2. Honorary President – R. Gilbert. President M. Ruzhansky.

Vice-Presidents (V.I. Burenkov – Europe, Africa; M. Yamamoto – Asia, Australia; Yongzhi Xu – Americas).

4.3. Main activity – biannual Congresses of the ISAAC, briefly Congresses on Analysis.

1997, Delaware, USA; 1999, Fukuoka, Japan;

2001, Berlin, Germany; 2003, Toronto, Canada;

2005, Catania, Italy; 2007, Ankara, Turkey; 2009, London, UK

4.4. NEXT ISAAC CONGRESS WILL BE HELD IN MOSCOW THROUGH 22-27 AUGUST 2011.

Around 500 participants are expected.

4.5. Among many sections there will be sections

Spaces of differentiable functions of several variables and applications,

Analytic and harmonic function spaces,

Approximation theory.

4.6. First Announcement will be distributed shortly and webpage

http://isaac2011.org

will be opened soon.

We consider the eigenvalue problem for the operator

$$Hu = (-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha \left(A_{\alpha\beta}(x) D^\beta u\right), \quad x \in \Omega,$$

subject to homogeneous Dirichlet or Neumann boundary conditions, where $m \in \mathbb{N}$, Ω is a bounded open set in $\mathbb{R}^N$ and the coefficients $A_{\alpha\beta}$ are real-valued Lipschitz continuous functions satisfying $A_{\alpha\beta} = A_{\beta\alpha}$ and the uniform ellipticity condition

$$\sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x)\xi_\alpha \xi_\beta \geq \theta |\xi|^2$$

for all $x \in \Omega$ and for all $\xi_\alpha \in \mathbb{R}, |\alpha| = m$, where $\theta > 0$ is the ellipticity constant.

We consider open sets Ω for which the spectrum is discrete and can be represented by means of a non-decreasing sequence of non-negative eigenvalues of finite multiplicity

$$\lambda_1[\Omega] \leq \lambda_2[\Omega] \leq \cdots \leq \lambda_n[\Omega] \leq \dots$$

Here each eigenvalue is repeated as many times as its multiplicity and

$$\lim_{n\to\infty} \lambda_n[\Omega] = \infty\,.$$

The aim is sharp estimates for the variation

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]|$$

of the eigenvalues corresponding to two open sets Ω_1, Ω_2.

There is vast literature on spectral stability problems for elliptic operators. However, very little attention has been devoted to the problem of spectral stability for higher order operators and in particular to the problem of finding explicit qualified estimates for the variation of the eigenvalues. Moreover, most of the existing qualified estimates for second order operators were obtained under certain regularity assumptions on the boundaries.

Our analysis comprehends

operators of arbitrary even order,

with homogeneous Dirichlet or Neumann boundary conditions,

and

open sets admitting arbitrarily strong degeneration.

We consider bounded open sets whose boundaries are just locally the subgraphs of continuous functions.

For any set V in $\mathbb{R}^N$ and $\delta > 0$

$$V_\delta = \{x \in V : \ d(x, \partial\Omega) > \delta\} .$$

Let $\rho > 0$, $s, s' \in \mathbb{N}$, $s' \leq s$ and $\{V_j\}_{j=1}^s$ be a family of bounded open cuboids and $\{r_j\}_{j=1}^s$ be a family of rotations in $\mathbb{R}^N$.
We say that that $\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ is an *atlas* in $\mathbb{R}^N$.

We denote by $C(\mathcal{A})$ the family of all open sets Ω in $\mathbb{R}^N$ satisfying the following properties:

(i) $\Omega \subset \bigcup_{j=1}^{s} (V_j)_\rho$ **and** $(V_j)_\rho \cap \Omega \neq \emptyset$;

(ii) $V_j \cap \partial\Omega \neq \emptyset$ **for** $j = 1, \dots s'$, $V_j \cap \partial\Omega = \emptyset$ **for** $s' < j \leq s$;

(iii) for $j = 1, ..., s$

$$r_j(V_j) = \{ x \in \mathbb{R}^N : \ a_{ij} < x_i < b_{ij}, \ i = 1, ..., N\},$$

and

$$r_j(\Omega \cap V_j) = \{x \in \mathbb{R}^N : \ a_{Nj} < x_N < g_j(\bar{x}), \ \bar{x} \in W_j\},$$

where $\bar{x} = (x_1, ..., x_{N-1})$**,** $W_j = \{\bar{x} \in \mathbb{R}^{N-1} : \ a_{ij} < x_i < b_{ij}, \ i = 1, ..., N-1\}$ **and** g_j **is a continuous function defined on** $\overline{W}_j$ **(it is meant that if** $s' < j \leq s$ **then** $g_j(\bar{x}) = b_{Nj}$ **for all** $\bar{x} \in \overline{W}_j$**);**

moreover for $j = 1, \ldots, s'$

$$a_{Nj} + \rho \leq g_j(\bar{x}) \leq b_{Nj} - \rho,$$

for all $\bar{x} \in \overline{W}_j$**.**

For all $\Omega_1, \Omega_2 \in C(\mathcal{A})$ **we define the** *atlas distance* $d_{\mathcal{A}}$ **by**

$$d_{\mathcal{A}}(\Omega_1, \Omega_2) = \max_{j=1,\ldots,s} \sup_{(\bar{x}, x_N) \in r_j(V_j)} |g_{1j}(\bar{x}) - g_{2j}(\bar{x})| \ .$$

We shall always assume that an atlas $\mathcal{A}$ *is fixed and all open sets* Ω *under consideration belong to* $C(\mathcal{A})$*.*

Theorem 1. Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$.

Let $m \in \mathbb{N}$, $L, \theta > 0$

For all $\alpha, \beta \in \mathbb{N}_0^N$ with $|\alpha| = |\beta| = m$, let $A_{\alpha\beta} \in C^{0,1}(\cup_{j=1}^s V_j)$ satisfy $A_{\alpha\beta} = A_{\beta\alpha}$,

$$\|A_{\alpha\beta}\|_{C^{0,1}(\cup_{j=1}^s V_j)} \leq L$$

and the ellipticity condition.

Then for each $n \in \mathbb{N}$ there exist $c_n, \epsilon_n > 0$ depending only on $n, N, \mathcal{A}, m, L, \theta$ such that for both Dirichlet and Neumann boundary conditions

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leq c_n d_{\mathcal{A}}(\Omega_1, \Omega_2),$$

for all $\Omega_1, \Omega_2 \in C(\mathcal{A})$ satisfying

$$d_{\mathcal{A}}(\Omega_1, \Omega_2) < \epsilon_n \, .$$

Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$. Let $\omega : [0,\infty[\to [0,\infty[$ be a continuous non-decreasing function such that $\omega(0) = 0$ and, for some $k > 0$, $\omega(t) \geq kt$ for all $0 \leq t \leq 1$.

Let $M > 0$. We denote by $C_M^{\omega(\cdot)}(\mathcal{A})$ the family of all open sets Ω in $\mathbb{R}^N$ belonging to $C(\mathcal{A})$ and such that all the functions g_j in the part (iii) of the definition of an open set of class $C(\mathcal{A})$ satisfy the condition

$$|g_j(\bar{x}) - g_j(\bar{y})| \leq M\omega(|\bar{x} - \bar{y}|),$$

for all $\bar{x}, \bar{y} \in \overline{W}_j$.

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Theorem 2. Under the assumptions of Theorem 1 for each $n \in \mathbb{N}$ there exist $c_n, \epsilon_n > 0$ depending only on $n, N, \mathcal{A}, m, L, M,$ θ, ω such that for both Dirichlet and Neumann boundary conditions

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leq c_n \omega(\epsilon),$$

for all $0 < \epsilon < \epsilon_n$ and for all $\Omega_1, \Omega_2 \in C_M^{\omega(\cdot)}(\mathcal{A})$ satisfying the inclusion

$$(\Omega_1)_\epsilon \subset \Omega_2 \subset (\Omega_1)^\epsilon$$

or the inclusion

$$(\Omega_2)_\epsilon \subset \Omega_1 \subset (\Omega_2)^\epsilon .$$

Theorem 3. Let $2 < p \le \infty$. Moreover, let $M_n > 0$, $n \in \mathbb{N}$.

Under the assumptions of Theorem 1 for each $n \in \mathbb{N}$ there exist $c_n, \epsilon_n > 0$ depending only on $n, N, \mathcal{A}, L, \theta, M_1, ..., M_n$ such that for both Dirichlet and Neumann boundary conditions

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \le c_n \Big(\mathrm{meas}\,(\Omega_1 \Delta \Omega_2)\Big)^{1-\frac{2}{p}},$$

where $\Omega_1 \Delta \Omega_2$ is the symmetric difference of Ω_1 and Ω_2, for all $\Omega_1, \Omega_2 \in C^{m-1,1}(\mathcal{A})$ satisfying

$$\|\varphi_k[\Omega_i]\|_{W^{m,p}(\Omega_i)} \le M_k\,, \quad i = 1, 2, \quad k = 1, ..., n$$

and

$$\mathrm{meas}\,(\Omega_1 \Delta \Omega_2) < \epsilon_n\,.$$

The exponent $1 - \frac{2}{p}$ is sharp.

Theorem 4. Let $m = 1$. Under the assumptions of Theorem 1 for each $n \in \mathbb{N}$ there exist $c, \epsilon_0 > 0$ depending only on $n, N, \mathcal{A}, L, \theta$ such that for the Dirichlet boundary conditions

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leq c \min\{\lambda_1[\Omega_1], \lambda_2[\Omega_2]\} \mathrm{meas}\,(\Omega_1 \Delta \Omega_2)\,,$$

for all $\Omega_1, \Omega_2 \in C^{1,1}(\mathcal{A})$ satisfying

$$\mathrm{meas}\,(\Omega_1 \Delta \Omega_2) < \epsilon_0\,.$$

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