INTEGRAL ESTIMATES OF DIFFERENTIABLE FUNCTIONS ON IRREGULAR DOMAINS

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Famous Sobolev embedding theorem $W_p^m(G) \subset L_q(G)$, characterized by the inequality

$$\|f|L_q(G)\| \leq C\|f|W_p^s(G)\| = C\left(\sum_{|\alpha|=s} \|D^\alpha f|L_p(G)\| + \|f\|L_p(G)\|\right), \quad (1)$$

$$s \in \mathbb{N}, \qquad 1 < p < q < \infty,$$

was established in 1938 (cf. [1]) for a domain $G \subset \mathbb{R}^n$ with cone condition for

$$s - \frac{n}{p} + \frac{n}{q} \geq 0. \qquad (2)$$

Relationship (2) (which defines the maximal possible value of q in theorem (1)) is also necessary condition for the embedding. This result by S.L. Sobolev was widened upon domains of more general kind: domains from classes $J_{\frac{n-1}{n}}$, $I_{p,\frac{1}{p}-\frac{1}{n}}$ (Maz'ya, 1960, 1975, cf. [2]), domains with John condition (John domain; Reshetnyak [3, 4]), domains with flexible cone condition (Besov, 1983, cf. [5]).

DEFINITION For $\sigma \geq 1$ the domain $G \subset \mathbb{R}^n$ is called *a flexible σ-cone condition domain*, if for some $T > 0$, $0 < \kappa_0 \leq 1$ for any $x \in G$ there exists a piece-wise smooth path

$$\gamma = \gamma_x : [0, T] \to G, \qquad \gamma(0) = x, \qquad |\gamma'| \leq 1 \quad \text{п.в.},$$

such that

$$\mathrm{dist}(\gamma(t), \mathbb{R}^n \setminus G) \geq \kappa_0 t^\sigma \qquad \text{при} \quad 0 < t \leq T.$$

In the case $\sigma = 1$ such a domain is also called a flexible cone condition domain. Domains which do not satisfy flexible cone condition will be called irregular.

For irregular domain which, in particular, may be an outer peak in the vicinity of some point, the embedding (1) may turn out to be not true for any parameters relationships or to be true only for some other stronger than (2) conditions binding n, s, p, q and depending also on geometrical propertitis of the domain G.

Maz'ya has indroduced classes of domains I_α (1960), $J_{p,\alpha}$ (1975), and established the embedding theorem (1) for $p = 1$ and, resp. for $p > 1$ when $s = 1$ with maximal possible q. Classes mentioned $I_\alpha, J_{p,\alpha}$ are defined in terms of isoperimetric or capacity inequalities.

In [6] it is shown, in particular, that for the flexible σ-cone condition domain the embedding (1) is valid for the following relationship between parameters

$$s - \frac{\sigma(n-1)+1}{p} + \frac{n}{q} \geq 0. \qquad (3)$$

This result for $s = 1$ belongs to Kilpelainen and Maly [7]. Labutin has established [8], that condition (3) is also *necessary* for this embedding. In [7], [6] one may find also weight generalizations of inequality (1) for flexible σ-cone.

Here we set forth some new integral representations of a function f via its derivatives $D^\alpha f, \alpha \in A, |\alpha| = s$, and obtain for them the pointwise estimates of a function through integral, which contain corresponding derivatives. Estimating the integral operators obtained in the same way as in [6], we obtain, in particular as an the sufficient conditions for validity of the inequality

$$\|f|L_q(G)\| \leq C \left(\sum_{\alpha\in A, |\alpha|=s} \|D^\alpha f|L_p(G)\| + \|f|L_p(G)\| \right) \tag{4}$$

for the domain G with flexible σ-cone condition. By means of examples it is shown that sufficient conditions obtained cannot be weakened.

The deduction of integral representation is accomplished for anisotropic case with respect to the smoothness, i.e. when the orders of the derivatives for different groups of variables may not necessarily coincide.
On its basis one can extend the embedding theorems onto the Sobolev spaces anisotropic with respect to smoothness..
It is possible to generalise the results obtained onto irregular domains of more general type as made by Trushin [9].

Let $\mathbb{N}$ be the set of all positive integers; $n \in \mathbb{N}$, $n \geq 2$;
$\mathbb{R}^n$ — n-dimensional Eucleadian space;
$1 \leq m \leq n, i_0 = 0, 1 \leq i_1 < i_2 < ... < i_m = n, n_j = i_j - i_{j-1}$,
$\chi_j : \{1, 2, ..., n\} \to \{0, 1\}$,

$$\chi_j(i) = \begin{cases} 0 & \text{при} \quad 1 \leq i \leq i_{j-1}, \\ 1 & \text{при} \quad i_{j-1} + 1 \leq i \leq i_j, \\ 0 & \text{при} \quad i_j + 1 \leq i \leq i_m = n. \end{cases}$$

For $x = (x_1, ..., x_n) \in \mathbb{R}^n, \alpha \in \mathbb{N}_0^n$ we put

$$x^j := \chi_j x := (0, \ldots, 0, x_{i_{j-1}+1}, \ldots, x_{i_j}, 0, \ldots, 0),$$

$$\alpha^j := \chi_j \alpha = (0, \ldots, 0, \alpha_{i_{j-1}+1}, \ldots, \alpha_{i_j}, 0, \ldots, 0),$$

so that $x = \sum\limits_{j=1}^{m} x^j$, $\alpha = \sum\limits_{j=1}^{m} \alpha^j$.

In the sequel $\delta \in (0,1)$ is small enough, область $G \subset \mathbb{R}^n$, $G_\delta = \{x \in G : \mathrm{dist}(x, \partial G) > \delta\}$,

$$\rho(x) = \mathrm{dist}(x, \mathbb{R}^n \setminus G), \quad \rho_1(x) = \min\{\rho(x), 1\},$$

$$B(x,R) = \{y : |y - x| < R\},$$

χ is a characteristic function of the ball or of the interval $B(0,1) \subset \mathbb{R}^k$, weight functions $v, w : G \to (0, \infty), u : G_\delta \to (0,\infty)$; u, v, w are locally summable. For the (measurable in a Lebesgue sense) set $E \subset \mathbb{R}^n$, $|E|_w = \int\limits_{E \cap G} w(x)\, dx$ and for $E \subset G$, $1 \leq p < \infty$

$$\|f\|_{p,E} = \|f|L_p(E)\|, \quad \|f|L_{p,v}(E)\| = \|fv^{\frac{1}{p}}|L_p(E)\|, \tag{5}$$

$$1 \leq p < q < \infty, \quad 1 \leq q, \quad s \in \mathbb{N}, \quad s - \frac{n}{p} + \frac{n}{q} \geq 0.$$

One of our goals is to find for the domain G with irregular boundary the validity conditions of the estimate

$$\|f|L_{q,w}(G)\| \leq C\left[\sum_{j=1}^{m} \sum_{\alpha=\alpha^j,|\alpha|=s} \|D^\alpha f|L_{p,v}(G)\| + \|f|L_{r,u}(G_\delta)\|\right], \quad (6)$$

based on the corresponding weak type estimate:

$$\sup_{\lambda>0} \lambda|\{x \in G : |f(x)| > \lambda\}|_w^{\frac{1}{q}} \leq$$

$$\leq C\left[\sum_{j=i}^{m} \sum_{\alpha=\alpha^j,|\alpha|=s} \|D^\alpha f|L_{p,v}(G)\| + \|f|L_{r,u}(G_\delta)\|\right]. \quad (7)$$

Let the domain $G \subset \mathbb{R}^n, 1 \leq m \leq n, s = (s_1, s_2, \ldots, s_m) \in \mathbb{N}^m$ and the function $f \in L(G, \mathrm{loc})$ posseses on G the Sobolev generalized derivatives

$$\{D^\alpha f : \alpha = \alpha^j, |\alpha| = s_j \, (j = 1, 2, \ldots, m)\}.$$

Let us construct the integral representation of the function f via given collection of derivatives. Consider the path

$$\Gamma(t, x) = (\Gamma_1(t, x), \ldots, \Gamma_n(t, x)), 0 \leq t \leq T, \tag{8}$$

where $\Gamma(0, x) = x$, functions $t \to \Gamma_i(t, x)$ are continuous and piece-wise continuously differentiable on $[0, T], |(\Gamma)'_t| \leq 1$.

The deduction of the integral representation of the function f at the point x is based on the construction of the averaging f_t of the function f over certain vicinity of the point $\Gamma(t,x)$ such that $f_t(x) \to f(x)$ as $t \to 0$ for a.a. $x \in G$ and $\frac{\partial}{\partial t} f_t$ is expressed via integrals, containing given derivatives of f. Then integral representation itself is obtained by applying Newton-Leibnitz formula, from which immediately follows the point-wise estimate

$$|f(x)| \leq \int_0^T \left| \frac{\partial}{\partial t} f_t(x) \right| dt + |f_T(x)|. \tag{9}$$

We suggest the construction of the averaging kernels first for the case $m = 1$ separately for $n = 1$ and for $n \geq 2$. The averaging kernel in general case $1 \leq m \leq n$ will be built as the product of these kernels. Let $n = 1$, $\mu \in C^\infty(\mathbb{R}^1)$,

$$\mu = \begin{cases} 0 & \text{при} \quad u \leq 0, \\ 1 & \text{при} \quad u \geq 1. \end{cases}$$

Let $k \in \mathbb{N}$. we put

$$\omega(u, \tau) := \frac{\partial^k}{\partial u} \left(\frac{u^{k-1}}{(k-1)!} \mu(u - \tau) \right), \tag{10}$$

so that $\int \omega(u, \tau) du = \mu(1) = 1$. Let $r : (0, T) \to (0, \infty)$ be a continuous and piece-wisw continuosly differentiable function, decreasing in some semi-vicinity of zero, $r(t) \to 0$ as $t \to 0$.

Let

$$f_t(x) = \int K_1(v, r(t), w(t)) f(x+v)\, dv, \tag{11}$$

$$K_1(v, r(t), w(t)) := \frac{1}{r(t)} \omega\left(\frac{v}{r(t)}, \frac{w(t)}{r(t)}\right), \tag{12}$$

where w is a continuous and piece-wise smooth function. Then

$$|K_1(v, r(t), w(t))| \le \frac{C}{r(t)} \left(\left|\frac{w(t)}{r(t)}\right|^{k-1} + 1\right) \chi\left(\frac{v - w(t)}{r(t)}\right), \tag{13}$$

$$\frac{\partial}{\partial t} K_1(v, r(t), w(t)) = \frac{\partial^k}{\partial v^k} \left\{ r(t)^{k-2} K\left(\frac{v}{r(t)}, \frac{w(t)}{r(t)}, r'(t), w'(t)\right)\right\}, \tag{14}$$

where

$$K(u, \tau, a, b) = \frac{u^{k-1}}{(k-1)!}[a(u-\tau) - b]\mu'(u-\tau),$$

so that

$$K(\cdot, \tau, a, b) \in C_0^\infty([-1+\tau, 1+\tau]),$$

$$|K(u, \tau, a, b)| \le C\chi(u-\tau)(1+|\tau|)^{k-1}(|a|+|b|). \tag{15}$$

Let now $n \geq 2, m = 1, s \in \mathbb{N}$. Let's consider that the function f is defined and has all necessary derivatives where it is needed for conducting the calculations. Let us take use of the averaging of f in the form of an integral of special linear combination of its values (cf. [11], [12]):

$$f_t(x) = \int \sum_{\nu=1}^{s} \sum_{k=0}^{s-1} a_\nu c_k(r_1(t)) f\big(x + \nu r(t) v + (\Gamma(t) - x)(1 + k r_1(t))\big) \eta(v)\, dv, \quad (16)$$

where $r : [0, t] \to [0, \infty]$ is a continuous and piece-wise smooth function., $r(t) > 0$ при $t > 0, r(t) \to 0$ при $t \to 0, |r'(t)| \leq c$, $|r(t)| \leq \frac{1}{2} \operatorname{dist}(\Gamma(t, x), \partial G)$, $r_1(t) = \frac{1}{4s} \frac{r(t)}{t}$, $\eta \in C_0^\infty\big(B\big(0, \frac{1}{4s}\big)\big)$, $a_j = (-1)^{j-1} \binom{s}{j}$,

$$\sum_{k=0}^{s-1} c_k(t) = 1, \qquad \sum_{k=0}^{s-1} c_k(t)(1 + kt)^l = 0 \quad \text{при} \quad l = 1, \ldots, s-1,$$

whence $|c_k(t)| \leq ct^{-s+1}, |c_k'(t)| \leq Ct^{-s}$.

From estimates [11], [12] by means of the Sobolev projectional decomposition of functions we conclude that

$$\left|\frac{\partial}{\partial t} f_t(x)\right| \leq C \sum_{|\alpha|=s} t^{s-1} r(t)^{-n} \int\limits_{|y-\Gamma(t,x)|<r(t)} |D^\alpha f(x+y)|\, dy.$$

By replacing of variables we rewrite f_t in the form

$$f_t(x) = \int K_n(y, r(t), \Gamma(t,x)) f(x+y)\, dy, \tag{17}$$

where

$$K_n(\cdot, r(t), \Gamma(t,x)) \in C_0^\infty(B(\Gamma(t,x), r(t))),$$

$$|K_n(y, r(t), \Gamma(t,x))| \leq C\chi\left(\frac{y-\Gamma(t,x)}{r(t)}\right) r(t)^{-s+1-n} t^{s-1}, \tag{18}$$

$$\int K_n(y, r(t), \Gamma(t,x)) dy = 1.$$

Let now $1 \leq m \leq n$, $s = (s_1, ..., s_m)$, $f \in L(G, \mathrm{loc}), x \in G, t > 0$. Consider the following averaging of f at the point x:

$$f_t = \int \prod_{j=1}^{m} K_{n_j}(y^j, r_j(t)\Gamma^j(t,x)) f(x+y)dy,$$

where $\Gamma^j(t,x) = (\Gamma_{i_{j-1}+1}(t,x), ..., \Gamma_{i_j}(t,x)), K_{n_j}$ is taken for $n_j \geq 2$ from (17), and for $n_j = 1$ from (11).
Differentiating with respect to t and taking into account (13)–(15), we obtain that

$$\left|\frac{\partial}{\partial t} f_t(x)\right| \leq C \sum_{i=1}^{m} \int \chi\left(\frac{y^j - \Gamma^j(t,x)}{r_j(t)}\right) \times$$

$$\times \prod_{j=1}^{m} r_j(t)^{-s_j+1-n_j} t^{s_j-1} r_i^{-n_i} t^{s_i-1} \sum_{\alpha=\alpha^i, |\alpha|=s_i} |D^\alpha f(x+y)|\, dy.$$

In the case $r_1(t) = ... = r_m(t) = r(t)$

$$\left|\frac{\partial}{\partial t} f_t(x)\right| \leq C \sum_{i=1}^{m} r(t)^{-\sum\limits_{1\leq j\leq m, j\neq i} s_j + m - 1} t^{\sum\limits_{j=1}^{m} s_j - m} r(t)^{-n} \times$$
$$\times \int \prod_{j=1}^{m} \chi\left(\frac{y^j - \Gamma^j(t,x)}{r(t)}\right) \sum_{\alpha=\alpha^i,\ |\alpha|=s_i} |D^\alpha f(x+y)|\, dy.$$

If $s_1 = ... = s_m = s$, then

$$\left|\frac{\partial}{\partial t} f_t(x)\right| \leq C r(t)^{-(s-1)(m-1)} t^{(s-1)m} r(t)^{-n} \int \prod_{j=1}^{m} \chi\left(\frac{y^j - \Gamma^j(t,x)}{r(t)}\right) \times$$
$$\times \sum_{i=1}^{m} \sum_{\alpha=\alpha^i, |\alpha|=s} |D^\alpha f(x+y)|\, dy.$$

The integral estimates established may be applied for obtaining embedding theorem of Sobolev space (generalised with the respect of the derivatives collection). We consider only the case of embedding

$$W^{\overline{s}}_{p,v,u}(G) \subset L_{q,w}(G), \quad 1 < p < q < \infty,$$

$$\overline{s} = (s_1, ..., s_m) \in \mathbb{N}^m, \quad 1 \leq m \leq n, \quad s_1 = ... = s_m = s.$$

Proof consists in getting the inequality of the type (4) with taking use of deduced poit-wise estimates of a function via integrals, containing the derivatives of a given funtion. The parameters of the integral representation have to corresponding geometrical properties of hte domain G.

The scheme of the reasonongs may be found in [6]. Therefore we set forth only one result for the domain G with flexible σ-cone condition ($\sigma \geq 1$).
THEOREM. Let $G \subset \mathbb{R}^n$ is a domain with flexible σ-cone condition ($\sigma \geq 1$),

$$s, m \in \mathbb{N}, 1 < p < q < \infty, 1 \leq r \leq q,$$

$$v(x) = \rho_1(x)^a, w(x) = \rho_1(x)^b, b \geq 0, a \geq 1 - n - (s-1)(m-1)p,$$

$$s - (s-1)(m-1)(\sigma - 1) - \frac{\sigma(n-1) + 1 + \sigma a}{p} + \frac{n+b}{q} \geq 0.$$

Then there is the constant $C > 0$, for which the estimate

$$\|f|L_{q,w}(G)\| \leq C \left[\sum_{i=1}^{n} \sum_{\alpha=\alpha^j, |\alpha|=s} \|D^\alpha f|L_{p,v}(G)\| + \|f|L_r(G_\delta)\| \right].$$

holds.

REMARK.

Condition (20) (at least for $a = b = 0$) cannot be weakened. This is proved by means of an example analogous to that by Labutin [8] for the case $m = 1$.

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